ELSEVIER

Journal of Geometry and Physics 44 (2002) 221–250

JOURNAL OF GEOMETRY AND PHYSICS

www.elsevier.com/locate/jgp

On complexes related with calculus of variations

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Received 30 May 2001; received in revised form 18 March 2002

Abstract

We consider the variational complex on infinite jet space and the complex of variational derivatives for Lagrangians of multidimensional paths and study relations between them. The discussion of the variational (bi)complex is set up in terms of a flat connection in the jet bundle. We extend it to supercase using a particular new class of forms. We establish relation of the complex of variational derivatives and the variational complex. Certain calculus of Lagrangians of multidimensional paths is developed. It is shown how covariant Lagrangians of higher order can be used to represent characteristic classes.

MSC: 58A20; 58A50; 58C50; 58A10; 70H03

Subj. Class.: Differential geometry

Keywords: Infinite jet space; Cartan connection; Variational complex; Variational derivatives; Forms on supermanifolds; Lagrangians; Multidimensional paths

1. Introduction

It is possible to include the “Euler–Lagrange operator” (the LHS of the Lagrange equations) into a complex on the space of infinite jets, the so-called variational complex. In this approach, Lagrangians (multiplied by volume forms) appear in a particular term of this complex, the other terms consisting of certain forms or classes of forms on the jet space. The explanation of this complex is in a spectral sequence due to Vinogradov. On the other hand, there is another complex, which we call the complex of variational derivatives (suggested by one of the authors), and in which the variational derivative is the main ingredient of the differential for *all* terms. Each term consists of Lagrangians of multidimensional paths, and the

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PII: S0393-0440(02)00075-X

differential increases the dimension of paths by one. The purpose of this paper is to initiate the study of the relation of these two complexes connected with variational problems.

The paper consists of three sections. In Sections 2 and 3 we mainly review the material concerning the variational complex. Obviously, it is well known to experts. However, we make a point in systematically using the framework of the canonical flat connection in the infinite jet bundle and following analogies with the more familiar differential-geometric constructions. The exposition is designed so as to make easy the transfer to the super case (which is done in Section 4). Interpreting the Cartan distribution (the contact distribution) as a connection works only for infinite jets. Hence, we make them, not finite jets, our primary objects. Hopefully, this exposition can help to straighten out and clarify some points.

In Section 2, we define the Cartan connection Γ in the jet bundle. It gives the exterior covariant differential on horizontal forms, which extends to the "horizontal" differential on arbitrary forms. The "vertical" differential is obtained as the Lie derivative along Γ considered as a form-valued vector field. This gives a canonical bicomplex on the jet space. In Section 3, Vinogradov's spectral sequence and variational complex are described in this language and the relation with the classical variational problem is explained.

In Section 4, we introduce the complex of variational derivatives. The variational complex is revisited and generalized for supermanifolds. Since the straightforward generalization of the usual bicomplex turns out to be not satisfactory, we introduce a new class of forms adequate for this case. These forms are particular hybrids of integral and differential forms. Then we establish relation between the complex of variational derivatives and the variational complex using a sequence of bicomplexes corresponding to increasing dimension of the manifolds of parameters. In the framework of the complex of variational derivatives we study covariant Lagrangians of the first and higher order, which are natural integration objects over surfaces. We begin to develop a "calculus of covariant Lagrangians", in particular, we consider the composition of Lagrangians and relate it with some densities for characteristic classes. This study is in progress and we hope to elaborate it elsewhere. Potentially it can be linked with symplectic reduction and various invariants for supermanifolds.

Main sources on jet geometry and variational complex are the books by Bocharov et al. [3] and Olver [10] (see also [11]). They contain plenty of reference for other works. Experts might notice that though our exposition in the preliminary sections owe much to these basic sources, our approach in many points differs from both [3,10]. In our work we use supermathematics. Not only we try to generalize to supercase, but we substantially rely on supermethods, which simplify constructions important for the geometry of jets. For supermanifolds theory we refer to [1,7], two chapters in [8], and to [18]. Concerning integration theory we particularly refer to [18,19].

2. Preliminaries: Cartan connection on infinite jet space

2.1. *Infinite jet space*

Consider a fibre bundle $\pi = \pi(E, M, F)$ over an r-dimensional manifold M with the fibre F.

Consider the fibre bundle $J^k(\pi) \to M$ of k-jets of local sections of E. If $x^a = (x^1, \dots, x^m)$ and $\varphi^\mu = (\varphi^1, \dots, \varphi^n)$ are some local coordinates on M and F respectively, then $(x^a, \varphi^\mu_\sigma) = (x^a, \varphi^\mu, \varphi^\mu_{a_1}, \varphi^\mu_{a_1 a_2}, \dots, \varphi^\mu_{a_1 \cdots a_k})$ are natural local coordinates on $J^k(\pi)$. Here $\sigma = a_1 \cdots a_p$ is a multi-index, $p = |\sigma| \le k$. Notice that the natural bundles $J^k(\pi) \to E$, $J^k(\pi) \to J^l(\pi)$ $(k > l)$ are not vector bundles, though their respective fibres are Euclidean spaces.

Consider the space $J^\infty(\pi)$ of infinite jets, i.e. the inverse limit of the manifolds $J^k(\pi)$. We denote by $C^\infty(J^k(\pi))$ the space of smooth functions on $J^k(\pi)$ $(k = 0, 1, 2, \dots, \infty)$. Every function f on $J^\infty(\pi)$, by the definition, has finite order, i.e. depends on a finite number of variables: $f = f(x, (\varphi^\mu, \dots, \varphi^\mu_{\alpha_1 \cdots \alpha_p}))$. In other words, $f \in C^\infty(J^p(\pi))$ for some finite p.

To every local section $s(x)$ of the fibre bundle E corresponds its k-jet, $(j_k s)(x) = (x^a, \varphi^\mu(x), \varphi^\mu_a(x), \dots, \varphi^\mu_{a_1 \cdots a_k}(x))$, where $\varphi^\mu_{a_1 \cdots a_p}(x) = \partial^p \varphi^\mu(x) / \partial x^{a_1} \cdots \partial x^{a_p}$, which is a section of $J^k(\pi)$, and the infinite jet $(js)(x) = (x^a, \varphi^\mu(x), \varphi^\mu_a(x), \dots, \varphi^\mu_{a_1 \cdots a_p}(x), \dots)$, which is a section of $J^\infty(\pi)$.

In the sequel we shall denote $J := J^\infty(\pi)$. Infinite jets will be simply called "jets". We shall use the notation $[\varphi]$ for the whole collection φ^μ_σ. The use of infinite jets is natural for the analysis of variational problems, because the variation of action increases the order of derivatives involved.

For a function $f = f(x, [\varphi])$ on J and a local section $s(x)$ of the bundle $E \to M$ the value $f|_s$ is defined as

$$f|_s = f \circ js = f(x, [\varphi(x)]). \tag{2.1}$$

This is a function on M.

Vectors and vector fields on the space of infinite jets J are derivations of the algebra $C^\infty(J)$. They have the form

$$X = X^a \frac{\partial}{\partial x^a} + X^\mu_\sigma \frac{\partial}{\partial \varphi^\mu_\sigma}, \tag{2.2}$$

where the number of non-zero coefficients can be infinite. (Summation in (2.2) is over multi-indices of all orders.) A vector on J is *vertical* if its projection on M vanishes, i.e. X^a in (2.2) equals to zero.

Consider now the algebra $\Omega = \Omega(J)$ of differential forms on the space of infinite jets. Differential forms that vanish on vertical vectors are called *horizontal* forms. They have the appearance $\omega = \sum \omega_{a_1 \cdots a_p}(x, \varphi^\mu_\sigma)\, \mathrm{d}x^{a_1} \cdots \mathrm{d}x^{a_1}$. (We don't write the wedge sign. Instead, we use a convention of supermathematics that the differential $\mathrm{d}x^a$ is odd for an even variable x^a.)

2.2. The Cartan connection

In the fibre bundle $J \to M$ there is a natural connection specified by the vector-valued one-form

$$\Gamma = \sum_{\mu, \sigma} \Gamma^\mu_\sigma \frac{\partial}{\partial \varphi^\mu_\sigma}, \tag{2.3}$$

which takes values in vertical vectors. (Summation in (2.3) goes over all multi-indices.) Here the coefficients

$$\Gamma^{\mu}_{\sigma} = \mathrm{d}\varphi^{\mu}_{\sigma} - \mathrm{d}x^{b}\varphi^{\mu}_{b\sigma} \tag{2.4}$$

are the so called Cartan one-forms. We call Γ the *Cartan connection*. Here $b\sigma$ is the multi-index $ba_1 \cdots a_k$, if σ is the multi-index $a_1 \cdots a_k$.

The distribution of the horizontal planes w.r.t. the connection Γ is called the *Cartan distribution*, its dimension is $m = \dim M$. The ideal in the algebra $\Omega(J)$ corresponding to the Cartan distribution (consisting of all forms vanishing on the distribution) will be denoted $C\Omega \subset \Omega$. It is generated by the Cartan forms (2.4). (Notice that the Cartan distribution is defined on $J^k(\pi)$ for finite k also, but there it does not correspond to a connection.)

The Cartan connection in the bundle $J \to M$ is flat. That means that the ideal $C\Omega$ is a differential ideal: $\mathrm{d}(C\Omega) \subset C\Omega$. The flatness of the Cartan connection (the integrability of the Cartan distribution) is an essential feature of the infinite jet space. For finite k the Cartan distribution on $J^k(\pi)$ is not integrable.

Every tangent vector on J can be uniquely decomposed into a vertical and a horizontal vector: $X = X_{\mathrm{vert}} + X_{\mathrm{hor}}$, where $X_{\mathrm{vert}} := \langle X, \Gamma\rangle$, the value of the one-form Γ on X, and $X_{\mathrm{hor}} := X - X_{\mathrm{vert}}$. X_{hor} belongs to the Cartan distribution, $\langle X_{\mathrm{hor}}, \Gamma\rangle = 0$:

$$X = X^{a}\frac{\partial}{\partial x^{a}} + X^{\mu}_{\sigma}\frac{\partial}{\partial\varphi^{\mu}_{\sigma}} = \underbrace{X^{a}\left(\frac{\partial}{\partial x^{a}} + \varphi^{\mu}_{a\sigma}\frac{\partial}{\partial\varphi^{\mu}_{\sigma}}\right)}_{\text{horizontal}} + \underbrace{(X^{\mu}_{\sigma} - X^{a}\varphi^{\mu}_{a\sigma})\frac{\partial}{\partial\varphi^{\mu}_{\sigma}}}_{\text{vertical}}. \tag{2.5}$$

Let us emphasize that every connection in an arbitrary fibre bundle defines two operations: a *covariant derivative of sections* and a *covariant derivative of functions on the total space*. If X is a vector on the base at a point x_0, then the covariant derivative of a section s along X, $\mathrm{D}_X s$, is the vertical component of the vector s_*X, which is a vertical tangent vector to the total space at the point $s(x_0)$; the covariant derivative along X of a function f on the total space at a point y_0 of the total space, $\mathrm{D}_X f$, is the usual derivative of f along the horizontal lift of the vector X to y_0. (In the familiar case of vector bundles, these notions correspond to the usual covariant derivative of sections, and that of sections of the dual bundle, which can be treated as fiberwise linear functions on the total space.) They are related by the "Leibniz formula":

$$X(s^* f) = (\mathrm{D}_X s) f + s^*(\mathrm{D}_X f), \tag{2.6}$$

where s is the section, f the function on the total space, and X is the vector field on the base. (Notice that then $\mathrm{D}_X s$ is a vector field along the map s.)

In particular, for the bundle $J \to M$ the covariant derivative of a section $j(x) = (x^a, \varphi^{\mu}(x), \varphi^{\mu}_{a}(x), \varphi^{\mu}_{ab}(x), \dots)$ is

$$\mathrm{D}_X j = \langle j_*(X), \Gamma\rangle = X^{a}\left(\frac{\partial\varphi^{\mu}_{\sigma}}{\partial x^{a}} - \varphi^{\mu}_{a\sigma}\right)\frac{\partial}{\partial\varphi^{\mu}_{\sigma}}. \tag{2.7}$$

The covariant derivative of a function f on J is

$$\mathrm{D}_X f = X^{a}\,\mathrm{D}_a f = X^{a}\left(\frac{\partial f}{\partial x^{a}} + \varphi^{\mu}_{a\sigma}\frac{\partial f}{\partial\varphi^{\mu}_{\sigma}}\right) \tag{2.8}$$

(in particular, $X(f|_s) = (\mathrm{D}_X f)|_s$), and there is a *covariant differential* of f as a horizontal 1-form:

$$\mathrm{D}f = \mathrm{d}x^a\,\mathrm{D}_a f = \mathrm{d}x^a\left(\frac{\partial f}{\partial x^a} + \varphi^\mu_{a\sigma}\frac{\partial f}{\partial \varphi^\mu_\sigma}\right). \tag{2.9}$$

The operation $\mathrm{D} : C^\infty(J) \to \Omega^1_{\mathrm{hor}}(J)$ extends to the *exterior covariant differential* on horizontal forms of arbitrary degree:

$$\mathrm{D}(\omega_{a_1\cdots a_k}\,\mathrm{d}x^{a_1}\cdots \mathrm{d}x^{a_k}) = (\mathrm{D}\omega_{a_1\cdots a_k})\,\mathrm{d}x^{a_1}\cdots \mathrm{d}x^{a_k}. \tag{2.10}$$

In a coordinate-free notation, $\mathrm{D}\omega = \mathrm{d}\omega - \Gamma\omega$, where

$$\Gamma\omega = \Gamma^\mu_\sigma \frac{\partial \omega_{a_1\cdots a_k}}{\partial \varphi^\mu_\sigma}\,\mathrm{d}x^{a_1}\cdots \mathrm{d}x^{a_k} \tag{2.11}$$

(this makes sense only for horizontal forms). We arrive at the sequence

$$0 \to \Omega^0_{\mathrm{hor}} \xrightarrow{\mathrm{D}} \Omega^1_{\mathrm{hor}} \xrightarrow{\mathrm{D}} \cdots \xrightarrow{\mathrm{D}} \Omega^{m-1}_{\mathrm{hor}} \xrightarrow{\mathrm{D}} \Omega^m_{\mathrm{hor}} \to 0, \tag{2.12}$$

where Ω^k_{hor} is the space of horizontal k-forms. Due to flatness of the Cartan connection, $\mathrm{D}^2 = 0$. Hence, (2.12) is a complex. This complex is the first part of the variational complex defined in the next section.

2.3. *The bicomplex $\Omega^{**}(J)$*

The operation (2.11) and the differential (2.10) introduced above can be extended from horizontal forms to the whole algebra $\Omega(J)$. This can be done using operations with form-valued vector fields. Let us briefly recall the necessary notions.

It is very convenient to use "super" language. A differential form on a manifold M can be considered as a function of even (commuting) variables x^a and odd (anticommuting) variables $\mathrm{d}x^a$: $\omega = \omega(x, \mathrm{d}x)$. We denote parity of all objects by tilde: $\widetilde{x^a} = 0$, $\widetilde{\mathrm{d}x^a} = 1$. The variables x^a, $\mathrm{d}x^a$ are coordinates on the supermanifold ΠTM associated with tangent bundle TM of M. In these terms the exterior differential d on $\Omega(M)$ is nothing but the vector field $\mathrm{d}x^i(\partial/\partial x^i) \in \mathrm{Vect}(\Pi TM)$.

A form-valued vector field $X \in \mathrm{Vect}(M, \Omega(M))$ has the appearance

$$X = X^i(x, \mathrm{d}x)\frac{\partial}{\partial x^i}. \tag{2.13}$$

(Warning: it is *not* a vector field on ΠTM.) The *Lie derivative* $\mathcal{L}_X$ *along* X is the vector field on ΠTM defined by the conditions: $\mathcal{L}_X f = Xf$ for an arbitrary function on M and $\mathcal{L}_X$ commutes with d:

$$[\mathcal{L}_X, \mathrm{d}] = \mathcal{L}_X \circ \mathrm{d} - (-1)^{\tilde{X}}\,\mathrm{d} \circ \mathcal{L}_X = 0. \tag{2.14}$$

Hence,

$$\mathcal{L}_X = X^i(x, \mathrm{d}x)\frac{\partial}{\partial x^i} + (-1)^{\tilde{X}}\,\mathrm{d}X^i(x, \mathrm{d}x)\frac{\partial}{\partial\, \mathrm{d}x^i}. \tag{2.15}$$

All vector fields on ΠTM commuting with d have the appearance $\mathcal{L}_X$ for some form-valued vector field X on M. The *Nijenhuis bracket* $[X, Y]$ of form-valued vector fields X and Y is defined by the following formula:

$$[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}. \tag{2.16}$$

It extends the usual commutator. Explicitly,

$$[X, Y] = (\mathcal{L}_X Y^i - (-1)^{\tilde{X}\tilde{Y}} \mathcal{L}_Y X^i) \frac{\partial}{\partial x^i}. \tag{2.17}$$

Now we can apply these constructions to our situation. The connection 1-form Γ can be viewed as a form-valued vector field on J:

$$\Gamma = \Gamma^\mu_\sigma \frac{\partial}{\partial \varphi^\mu_\sigma} = (\mathrm{d}\varphi^\mu_\sigma - \mathrm{d}x^a \varphi^\mu_{a\sigma}) \frac{\partial}{\partial \varphi^\mu_\sigma}. \tag{2.18}$$

The flatness of Γ is equivalent to

$$[\Gamma, \Gamma] = 0 \tag{2.19}$$

(Nijenhuis bracket). It follows that

$$[\mathcal{L}_\Gamma, \mathcal{L}_\Gamma] = 2\mathcal{L}^2_\Gamma = 0 \tag{2.20}$$

for the Lie derivative along Γ. From the explicit formula above, we get for an arbitrary form $\omega \in \Omega(J)$:

$$\mathcal{L}_\Gamma \omega = (\mathrm{d}\varphi^\mu_\sigma - \mathrm{d}x^a \varphi^\mu_{a\sigma}) \frac{\partial \omega}{\partial \varphi^\mu_\sigma} - \mathrm{d}x^a\, \mathrm{d}\varphi^\mu_{a\sigma} \frac{\partial \omega}{\partial\, \mathrm{d}\varphi^\mu_\sigma}. \tag{2.21}$$

Hence, $\mathcal{L}_\Gamma x^a = 0$, $\mathcal{L}_\Gamma \varphi^\mu_\sigma = \Gamma^\mu_\sigma$, $\mathcal{L}_\Gamma\, \mathrm{d}x^a = 0$, $\mathcal{L}_\Gamma\, \mathrm{d}\varphi^\mu_\sigma = -\mathrm{d}x^a\, \mathrm{d}\varphi^\mu_{a\sigma}$. Notice that for horizontal forms $\mathcal{L}_\Gamma \omega = \Gamma\omega$ as defined in (2.11). It also follows that

$$\mathcal{L}_\Gamma \Gamma^\mu_\sigma = 0, \tag{2.22}$$

because $[\Gamma, \Gamma] = 2(\mathcal{L}_\Gamma \Gamma^\mu_\sigma)(\partial/\partial\varphi^\mu_\sigma)$. It turns out to be convenient to express forms in terms of the variables x^a, φ^μ_σ, $\mathrm{d}x^a$, Γ^μ_σ instead of x^a, φ^μ_σ, $\mathrm{d}x^a$, $\mathrm{d}\varphi^\mu_\sigma$. Written in these variables, $\mathcal{L}_\Gamma$ is simply

$$\mathcal{L}_\Gamma = \Gamma^\mu_\sigma \frac{\partial}{\partial \varphi^\mu_\sigma} \tag{2.23}$$

for all forms.

Define on arbitrary forms operations δ and D so that $\delta = \mathrm{D} + \delta$:

$$\delta\omega := \mathcal{L}_\Gamma \omega, \tag{2.24}$$

$$\mathrm{D}\omega := \mathrm{d}\omega - \mathcal{L}_\Gamma \omega. \tag{2.25}$$

The operators δ and D are called the *vertical* and *horizontal differentials*, respectively. (Quite often the notation d_V and d_H is used.) As $\delta^2 = \mathcal{L}^2_\Gamma = 0$ and $[\mathrm{d}, \delta] = [\mathrm{d}, \mathcal{L}_\Gamma] = 0$, hence $\mathrm{D}^2 = 0$, and $[\mathrm{D}, \delta] = \mathrm{D} \circ \delta + \delta \circ \mathrm{D} = 0$, so we have a bicomplex. Clearly, the horizontal differential D extends the covariant differential of horizontal forms (2.10).

We can introduce an invariant *bigrading* in the algebra $\Omega(J)$ by the degrees in the variables $\mathrm{d}x^a$ and Γ^μ_σ. By definition,

$$\Omega^{p,q}(J) := \{\omega \in \Omega(J) | \#\Gamma^\mu_\sigma = p \text{ and } \#\mathrm{d}x^a = q\}, \tag{2.26}$$

where $\#\mathrm{d}x^a$, etc. means the degree in the respective variables. The total degree is $p+q$. Since one can easily obtain $\mathrm{D}(x^a) = \mathrm{d}x^a$, $\mathrm{D}(\varphi^\mu_\sigma) = \mathrm{d}x^a \varphi^\mu_{a\sigma}$, $\mathrm{D}(\mathrm{d}x^a) = 0$, $\mathrm{D}(\Gamma^\mu_\sigma) = \mathrm{d}x^a \Gamma^\mu_{a\sigma}$ (from the formulas for d and for $\delta = \mathcal{L}_\Gamma$ above), we have for a form $\omega = \omega(x, [\varphi], \mathrm{d}x, \Gamma)$:

$$\mathrm{D}\omega = \mathrm{d}x^a \left(\frac{\partial \omega}{\partial x^a} + \varphi^\mu_{a\sigma} \frac{\partial \omega}{\partial \varphi^\mu_\sigma} + \Gamma^\mu_{a\sigma} \frac{\partial \omega}{\partial \Gamma^\mu_\sigma} \right), \tag{2.27}$$

$$\delta\omega = \Gamma^\mu_\sigma \frac{\partial \omega}{\partial \varphi^\mu_\sigma}. \tag{2.28}$$

Hence, $\mathrm{D} : \Omega^{p,q}(J) \to \Omega^{p,q+1}(J)$, $\delta : \Omega^{p,q}(J) \to \Omega^{p+1,q}(J)$.

(There is a remote analogy with complex manifolds, where the integrability is provided by the condition $[J, J] = 0$ and there is a bigrading of forms by $\mathrm{d}z$ and $\mathrm{d}\bar{z}$. Of course, in our case there is no such symmetry that exists between the holomorphic and antiholomorphic differentials.)

2.4. Evolutionary vector fields

Consider the action of vector fields on the connection 1-form Γ. Notice first that geometrically Γ has the meaning of a projector onto the vertical subspace in a tangent space to J, the kernel of this projection being the horizontal subspace (thus defined). It is easy to see that for an arbitrary projector P, its preservation by some flow is equivalent to the preservation of two distributions, the image of P and the kernel of P (the image of $1 - P$). The preservation of the image as such is equivalent to $(1 - P) \circ \mathcal{L}_X P = 0$, while the preservation of the kernel is equivalent to $P \circ \mathcal{L}_X P = 0$. So in our case, vector fields that preserve Γ, preserve both the Cartan distribution and the fibre structure in $J \to M$. Preservation of the Cartan distribution as such is equivalent to $\langle \mathcal{L}_X \Gamma, \Gamma \rangle = 0$.

As the Lie derivative $\mathcal{L}_X \Gamma$ is nothing but the Nijenhuis bracket $[X, \Gamma]$, it follows that at no extra cost we can include in our consideration vector fields taking values in forms. (This might be useful, e.g. for studying deformations.) Then $[X, \Gamma] = -(-1)^{\tilde{X}} [\Gamma, X]$.

We decompose the tangent space at some point $j \in J$ into the horizontal subspace with a natural basis $\mathrm{D}_a = \partial/\partial x^a + \varphi^\mu_{a\sigma} \partial/\partial \varphi^\mu_\sigma$ and the vertical subspace with a natural basis $\partial/\partial \varphi^\mu_\sigma$. Since $\langle \mathrm{D}_a, \mathrm{d}x^b \rangle = \delta^b_a$, $\langle \mathrm{D}_a, \Gamma^\mu_\sigma \rangle = 0$, $\langle \partial/\partial \varphi^\mu_\sigma, \mathrm{d}x^b \rangle = 0$, $\langle \partial/\partial \varphi^\mu_\sigma, \Gamma^\nu_\tau \rangle = \delta^\sigma_\tau \delta^\nu_\mu$, this is the dual basis for the basis of 1-forms $\mathrm{d}x^a$, Γ^μ_σ. It makes sense to consider horizontal and vertical vector fields separately.

Clearly, *horizontal vector fields on J are tangent to the Cartan distribution and preserve it automatically. They preserve also the bundle structure $J \to M$ if they have the form* $\mathrm{D}_X = X^a(x)\,\mathrm{D}_a$ *where* $X^a = X^a(x)$ (*i.e., if they have the form of the covariant derivatives along vector fields on M*). For the general form-valued case, the condition is $\delta X^a = 0$ componentwise. Indeed, one can check that $\mathcal{L}_X \Gamma^\mu_\sigma = X^a \Gamma^\mu_{a\sigma}$, hence $\mathcal{L}_X \Gamma = [X, \Gamma] = -(-1)^{\tilde{X}} \delta X^a\, \mathrm{D}_a$, by a direct computation with the formula (2.17). The complex of form-valued horizontal vector fields obtained here:

$$\Omega^{*,q}(J, \mathrm{Vect}_{\mathrm{hor}}) \xrightarrow{\delta} \Omega^{*+1,q}(J, \mathrm{Vect}_{\mathrm{hor}}) \tag{2.29}$$

with δ applied componentwise, is an example of "vertical" complexes of forms on J with coefficients in arbitrary vector bundles coming from M, i.e. with δ-flat transition functions (independent of φ^μ_σ).

The case of vertical vector fields is more interesting. Consider a section of the pull-back to J of the vertical subbundle of TE. It has an appearance of a "generalized" vertical vector field on the bundle E with coefficients depending on jets:

$$Y = Y^\mu(x, [\varphi]) \frac{\partial}{\partial \varphi^\mu}. \tag{2.30}$$

Denote by $\mathcal{P}_Y$ the vertical vector field on J uniquely defined by the conditions: $\mathcal{P}_Y f = Yf$ for all functions on E and $\mathcal{L}_{\mathcal{P}_Y} \Gamma = 0$. Explicitly,

$$\mathcal{P}_Y = \sum_{|\sigma| \geq 0} \mathrm{D}^{|\sigma|}_\sigma Y^\mu \frac{\partial}{\partial \varphi^\mu_\sigma}. \tag{2.31}$$

Objects (2.30) (to which correspond genuine vector fields $\mathcal{P}_Y$ on J) are called *evolutionary vector fields* [10]. *Vertical vector fields on J preserve the Cartan distribution* (they clearly preserve the bundle structure in $J \to M$) *if they have the form $\mathcal{P}_Y$ for some evolutionary vector field Y*.

Indeed, for an arbitrary vertical vector field taking values in forms

$$\hat{Y} = Y^\mu_\sigma(x, [\varphi], \mathrm{d}x, \Gamma) \frac{\partial}{\partial \varphi^\mu_\sigma} \tag{2.32}$$

one can directly find that $\mathcal{L}_{\hat{Y}} \Gamma^\mu_\sigma = (-1)^{\tilde{Y}} (\mathrm{d}Y^\mu_\sigma - \mathrm{d}x^a Y^\mu_{a\sigma})$, hence

$$\mathcal{L}_{\hat{Y}} \Gamma = [\hat{Y}, \Gamma] = (-1)^{\tilde{Y}} \, \mathrm{d}x^a (\mathrm{D}_a Y^\mu_\sigma - Y^\mu_{a\sigma}) \frac{\partial}{\partial \varphi^\mu_\sigma}. \tag{2.33}$$

For a usual vector field, we deduce that the condition $\mathcal{L}_Y \Gamma = \langle \mathcal{L}_Y \Gamma, \Gamma \rangle = 0$ gives rise to the inductive formula

$$Y^\mu_{a\sigma} = \mathrm{D}_a Y^\mu_\sigma, \tag{2.34}$$

hence all coefficients Y^μ_σ are uniquely defined from the coefficient $Y^\mu = Y^\mu(x, [\varphi])$ as $Y^\mu_\sigma = \mathrm{D}^{|\sigma|}_\sigma Y^\mu = \mathrm{D}_{a_1} \cdots \mathrm{D}_{a_{|\sigma|}} Y^\mu$. The initial term $Y^\mu(\partial/\partial\varphi^\mu)$ corresponds to the restriction of $\hat{Y}$ to the subalgebra of functions of x^a, φ^μ. (The formula $[\Gamma, Y] = -(-1)^{\tilde{Y}}[Y, \Gamma] = (\mathrm{d}x^a Y^\mu_{a\sigma} - \mathrm{D}Y^\mu_\sigma)(\partial/\partial\varphi^\mu_\sigma)$ can be interpreted as a differential in a complex of form-valued vertical vector fields.)

Since the vertical vector fields on J preserving the Cartan distribution form a subalgebra in $\mathrm{Vect}(J)$, the formula

$$[\mathcal{P}_Y, \mathcal{P}_Z] = \mathcal{P}_{[Y,Z]} \tag{2.35}$$

defines a bracket $[Y, Z]$ on evolutionary vector fields, known as the *Jacobi bracket* [3,10], which extends the usual commutator of vertical fields on E. (Compare with the construction of the Nijenhuis bracket.) Explicitly,

$$[Y, Z] = (\mathcal{P}_Y Z^\mu - \mathcal{P}_Z Y^\mu)\frac{\partial}{\partial \varphi^\mu}. \tag{2.36}$$

Notice the following useful properties of evolutionary fields:

$$[\mathrm{D}, \iota_{\mathcal{P}_Y}] = 0, \tag{2.37}$$

$$[\delta, \iota_{\mathcal{P}_Y}] = \mathcal{L}_{\mathcal{P}_Y}, \tag{2.38}$$

where ι stands for the interior multiplication. The equality (2.37) follows from (2.27) and (2.31), and the equality (2.38) follows then from (2.37) and the usual relation between d and ι. It also follows that

$$[\mathcal{L}_{\mathcal{P}_Y}, \mathrm{D}] = 0 \tag{2.39}$$

for all evolutionary fields Y.

Every vector field on J preserving the Cartan connection Γ can be uniquely decomposed into $\mathrm{D}_X + \mathcal{P}_Y$, where X is a vector field on M and Y is an evolutionary vector field. If we do not require that the bundle structure $J \to M$ is preserved, then D_X can be replaced by an arbitrary horizontal field. It is easy to see that horizontal fields form an ideal in the Lie algebra of vector fields preserving the Cartan distribution, and $[\mathrm{D}_X, \mathcal{P}_Y] = 0$ for all $X \in \mathrm{Vect}(M)$.

Evolutionary vector fields model the variations of the independent argument of the classical calculus of variations, see below in Section 3.

Example 2.1. Every vector field on the total space E,

$$X = X^a(x, \varphi)\frac{\partial}{\partial x^a} + X^\mu(x, \varphi)\frac{\partial}{\partial \varphi^\mu}, \tag{2.40}$$

defines the Lie derivative (infinitesimal variation) of a section $s : M \to E$:

$$(\mathcal{L}_X s)(x) = \left(X^a(s(x))\frac{\partial \varphi^\mu}{\partial x^a}(x) - X^\mu(s(x))\right)\frac{\partial}{\partial \varphi^\mu}, \tag{2.41}$$

which is a vector field along the map $s : M \to E$. We can reinterpret it as a particular evolutionary vector field:

$$Y := (X^a(x, \varphi)\varphi^\mu_a - X^\mu(x, \varphi))\frac{\partial}{\partial \varphi^\mu}. \tag{2.42}$$

The subalgebra of such fields under the Jacobi bracket is isomorphic to the Lie algebra $\mathrm{Vect}(E)$.

For every vector field (2.40) on E, its *prolongation* $X^{(\infty)} \in \mathrm{Vect}(J)$ can be defined by the conditions that $X^{(\infty)} f = Xf$ for all functions on E and $X^{(\infty)}$ preserves the Cartan distribution. Decomposing $X^{(\infty)}$ into the horizontal and vertical components, we see that

$X^{(\infty)} = X^a(x, \varphi)\mathrm{D}_a - \mathcal{P}_Y$ where Y is the evolutionary vector field (2.42) corresponding to X. (The minus sign before $\mathcal{P}_Y$ is explained by a "duality" of functions and sections, hence the opposite signs in the respective Lie derivatives.)

3. Vinogradov's spectral sequence and variational complex

3.1. Main construction

In the complex (Ω, d) of differential forms on infinite jet space J there is a natural filtration by the powers of the Cartan ideal $C\Omega$:

$$\cdots \subset C^k\Omega \subset C^{k-1} \subset \cdots \subset \Omega \subset C\Omega \subset C^0\Omega = \Omega, \tag{3.1}$$

where $C^k\Omega$ denotes the kth power of the ideal $C\Omega$ and d is the usual de Rham differential. $\mathrm{d}(C^k\Omega) \subset C^k\Omega$, because the Cartan distribution is integrable. By standard homological algebra, we come to the *Vinogradov spectral sequence* [14],

$$(E_r^{**}, \mathrm{d}_r) \Rightarrow H^*_{\mathrm{DR}}(J) \tag{3.2}$$

converging to the de Rham cohomology of the space J, with the zeroth term

$$E_0^{p,q} = \frac{C^p\Omega^{p+q}}{C^{p+1}\Omega^{p+q}}. \tag{3.3}$$

In our case of a bundle $E \to M$ (see Remark 3.1 below) the filtration (3.1) is actually induced by the bigrading (2.26):

$$C^p\Omega^{p+q} = \Omega^{p,q} \oplus \Omega^{p+1,q-1} \oplus \Omega^{p+2,q-2} \oplus \cdots, \tag{3.4}$$

and the Vinogradov spectral sequence is a spectral sequence of the bicomplex $(\Omega^{**}(J), \mathrm{D}, \delta)$. We can identify its zeroth term (3.3) with Ω^{**}:

$$E_0^{p,q} = \Omega^{p,q}(J), \tag{3.5}$$

and the differential d_0 is the horizontal differential $\mathrm{D} : \Omega^{p,q} \to \Omega^{p,q+1}$. The zeroth row of E_0^{**} is the complex (2.12) of horizontal differential forms on J.

Clearly, $\Omega^{p,q} = 0$ for $q \geq m+1$ (m is the dimension of the base M). Since the fibres of the bundle $J \to E$ are contractible, we may say that the spectral sequence converges to $H^*(E)$. The following essential fact holds:

Proposition 3.1. *If $p \geq 1$, then for all $q < m$*

$$E_1^{p,q} = H^q(\Omega^{p,*}, \mathrm{D}) = 0. \tag{3.6}$$

Hence, $E_1^{0,q} = E_\infty^{0,q} = H^q(E)$ for $q \leq m-1$, and $E_2^{p,m} = E_\infty^{p,m} = H^{p+m}(E)$ for $p \geq 0$.

This fact has a purely algebraical origin [3]. We shall illustrate the main idea of the proof of this proposition when analyzing the content of the space $E_1^{1,m}$ (see below).

Because of the degeneration property (3.6), all the information about the spectral sequence (3.2) is contained in the following complex [3]:

$$E_0^{0,0} \xrightarrow{d_0} E_0^{0,1} \xrightarrow{d_0} \cdots \xrightarrow{d_0} E_0^{0,m-1} \xrightarrow{d_0} E_0^{0,m} \xrightarrow{d_1 \circ p} E_1^{1,m} \xrightarrow{d_1} E_1^{2,m} \xrightarrow{d_1} E_1^{3,m} \xrightarrow{d_1} \cdots, \tag{3.7}$$

where p is the projection $E_0^{0,m} \xrightarrow{p} E_1^{0,m}$. This complex is called the *variational complex*.

Two halves of the complex (3.7) can be described in terms of the bicomplex $\Omega^{**}(J)$ as follows. The first half is the complex of horizontal forms (2.12): $E_0^{0,q} = \Omega^{0,q}$ with $d_0 = D$; the second half consists of classes of forms: $E_1^{p,m} = \Omega^{p,m}/D(\Omega^{p,m-1})$, and d_1 is the vertical differential δ acting on classes.

Hence, the cohomology of the complex (3.7) in dimensions $0, \ldots, m-1$ coincides with $E_1^{0,k}$, $k = 0, \ldots, m-1$. In dimensions $k \geq m$ the cohomology coincides with $E_2^{0,k}$. It follows that the cohomology of the variational complex in all dimensions is exactly $H^*_{DR}(J) = H^*_{DR}(E)$. In particular, the variational complex is acyclic after $(m+n)$-th term (where n is the dimension of the fibre).

Remark 3.1. In a more general case of the so-called "projective jets" [3,10] for an $(m+n)$-dimensional manifold E, without a bundle structure, there is no bicomplex, but the filtration (3.1) and the Vinogradov spectral sequence survive. The space of projective jets is not contractible to E.

3.2. Relation with the classical variational problem

The first part of the variational complex (3.7) is the complex of horizontal differential forms (2.12). Horizontal forms of top degree (m-forms) $\mathbf{L} = L(x, [\varphi])\mathcal{D}x$ are *Lagrangians*. We denote by $\mathcal{D}x$ the coordinate volume form $dx^1 \cdots dx^m$. Later we shall also use the notation $\mathcal{D}x_a$ for the $(m-1)$-form $(-1)^{a-1}\, dx^1 \cdots dx^{a-1}\, dx^{a+1} \cdots dx^m$. If $s(x) = (x, \varphi(x))$ is a section of the fibre bundle $E \to M$, then the value of the Lagrangian on this section $\mathbf{L}|_s = L|_s \mathcal{D}x$ (see (2.1)) defines a top-degree differential form (m-form) on the manifold M.

Hence, a Lagrangian $\mathbf{L}$ defines an action functional on sections of E:

$$S[\varphi] = \int_M \mathbf{L} = \int_M L\left(x, \varphi(x), \frac{\partial \varphi}{\partial x}(x), \ldots\right) \mathcal{D}x \tag{3.8}$$

Solution of the variational problem for this functional leads to the Euler–Lagrange equations for the section $s(x)$:

$$\mathcal{F}_\mu|_s = 0, \tag{3.9}$$

where the *variational derivative* $\mathcal{F}_\mu = \mathcal{F}_\mu(\mathbf{L})$ is defined by the following expression:

$$\mathcal{F}_\mu(\mathbf{L}) = \frac{\partial L}{\partial \varphi^\mu} - D_a \frac{\partial L}{\partial \varphi^\mu_a} + D^2_{ab} \frac{\partial L}{\partial \varphi^\mu_{ab}} - \cdots. \tag{3.10}$$

The map $E_0^{0,m} \xrightarrow{p} E_1^{0,m}$ used in the construction of (3.7) corresponds to the projection of Lagrangians to equivalence classes modulo D-coboundaries, $\mathbf{L} \mapsto [\mathbf{L}] = \mathbf{L} + D(\Omega^{0,m-1})$.

If $\mathbf{F} = F^a\mathcal{D}x_a$ is a horizontal $(m-1)$-form, then $\mathrm{D}\mathbf{F} = \mathrm{D}_a F^a\mathcal{D}x$, so, in the classical language, equivalent Lagrangians "differ by a divergence". On arbitrary section s,

$$\mathbf{L}'|_s - \mathbf{L}|_s = \mathrm{D}\mathbf{F}|_s = \frac{\partial}{\partial x^a}(F^a|_s)\mathcal{D}x. \tag{3.11}$$

Hence, $\int \mathbf{L}'$ and $\int \mathbf{L}$ may differ only by boundary terms, and the variational derivative (3.10) is well-defined on classes $[\mathbf{L}] \in E_1^{0,m}$.

Now compare these classical considerations with the corresponding differential in the variational complex (3.7). Consider the action of the differential $\mathrm{d}_1 \circ p : E_0^{0,m} \to E_1^{1,m}$, i.e. the action of the differential d_1 on the equivalence class $[\mathbf{L}] \in E_1^{0,m}$ of a Lagrangian $\mathbf{L} = L\mathcal{D}x$. According to (2.24),

$$\mathrm{d}_1[\mathbf{L}] = [\delta\mathbf{L} + \cdots] = \left[\sum_{\sigma,\mu} \varGamma_\sigma^\mu \frac{\partial L}{\partial \varphi_\sigma^\mu}\mathcal{D}x + \cdots\right], \tag{3.12}$$

where dots denote differential forms belonging to the image of the differential $\mathrm{D} : \varOmega^{1,m-1} \to \varOmega^{1,m}$. For example, for $\mathbf{F} = F^\alpha\mathcal{D}x_\alpha \in \varOmega^{0,m-1}$, we have $\mathrm{d}_1[\mathrm{D}\mathbf{F}] = [\delta\,\mathrm{D}\mathbf{F}] = -[\mathrm{D}\delta\mathbf{F}] = 0$ (compare with (3.11)).

To find the correspondence between the image (3.12) of the differential $\mathrm{d}_1 \circ p$ and the variational derivative (3.10), we shall study the content of the space $E_1^{1,m}$ in (3.7).

Consider the filtration in the Cartan ideal $C\varOmega$ induced by the order of the multi-index of Cartan forms. If $\varOmega^{1,q}$ is the space of differential $(q+1)$-forms that are linear in Cartan forms, then denote by $\varOmega_{(k)}^{1,q}$ the subspace of $\varOmega^{1,q}$ consisting of forms $\omega = \sum_{|\sigma|\le k} \varGamma_\sigma^\mu \omega_\mu^\sigma$, where ω_μ^σ is a horizontal q-form. One can show that if $\omega \in \varOmega_{(k)}^{1,q}$ for $k \ge 1$ and $\mathrm{D}\omega = 0$, then ω is equal up to a D-coboundary to a form $\omega' = \omega - \mathrm{D}\tau \in \varOmega_{(k-1)}^{1,q}$. Thus every D-closed $\omega \in \varOmega_{(k)}^{1,q}$ is equivalent to a form $\tilde{\omega} \in \varOmega_{(0)}^{1,q}$. For example, if $\omega \in \varOmega_{(k)}^{1,m}$, $\omega = \varGamma_\sigma^\mu B_\mu^\sigma(x,[\varphi])\mathcal{D}x$ with $\sigma = a_1 \cdots a_k$, then automatically $\mathrm{D}\omega = 0$ and from (2.27) and (2.28) it follows that

$$\begin{aligned}\omega &= \varGamma_\sigma^\mu B_\mu^\sigma \mathcal{D}x = \delta\varphi_\sigma^\mu B_\mu^\sigma \mathcal{D}x = (\delta \mathrm{D}\varphi_{\sigma'}^\mu) B_\mu^{a\sigma'}\mathcal{D}x_a = -(\mathrm{D}\delta\varphi_{\sigma'}^\mu) B_\mu^{a\sigma'}\mathcal{D}x_a \\ &= -\mathrm{D}(\delta\varphi_{\sigma'}^\mu B_\mu^{a\sigma'}\mathcal{D}x_a) - \delta\varphi_{\sigma'}^\mu \mathrm{D}(B_\mu^{a\sigma'}\mathcal{D}x_a) = -\varGamma_{\sigma'}^\mu(\mathrm{D}_a B_\mu^{a\sigma'})\mathcal{D}x - \mathrm{D}(\delta\varphi_{\sigma'}^\mu B_\mu^{a\sigma'}\mathcal{D}x_a),\end{aligned}$$

where $\sigma' = a_2 \cdots a_k$. By iterating this we come to the map $\rho : \varOmega^{1,m} \to \varOmega_{(0)}^{1,m}$:

$$\rho : \varGamma_\sigma^\mu B_\mu^\sigma \mathcal{D}x \mapsto \varGamma^\mu (-1)^{|\sigma|}\mathrm{D}_\sigma^{|\sigma|} B_\mu \mathcal{D}x, \tag{3.13}$$

so that

$$\rho(\omega) = \omega + \mathrm{D}(\varOmega^{1,m-1}). \tag{3.14}$$

Proposition 3.2. *The map ρ defined by (3.13) establishes an isomorphism between the space $E_1^{1,m}$ and the subspace $\varOmega_{(0)}^{1,m} \subset \varOmega^{1,m}$.*

We call forms $\omega \in \varOmega_{(0)}^{1,m}$ the *canonical representatives* of cohomological classes $[\omega] \in E_1^{1,m}$. (If $\omega \in \varOmega_{(0)}^{1,q}$, where $q < m$, and $\mathrm{D}\omega = 0$, then one can notice directly that $\omega = 0$.

Thus we come to the statement of Proposition 3.1 in the case $q = 1$. For $p \geq 2$ the argument is similar.) Similar analysis of the contents of $E_1^{p,m}$ can be performed for $p > 1$.

Using the homomorphism ρ, we immediately deduce from (3.12) that

$$\mathrm{d}_1[\mathbf{L}] = [\delta\mathbf{L}] = \left[\sum_{\sigma,\mu} \Gamma_\sigma^\mu \frac{\partial L}{\partial \varphi_\sigma^\mu} \mathcal{D}x\right] = [\Gamma^\mu \mathcal{F}_\mu(\mathbf{L})\mathcal{D}x], \tag{3.15}$$

where $\mathcal{F}_\mu(\mathbf{L})$ is the variational derivative (3.10) for the Lagrangian $\mathbf{L} = L\mathcal{D}x$.

Relations between these purely algebraic considerations and the variational problem for the functional (3.8) can be established with the help of evolutionary vector fields considered above. For an evolutionary field $Y = Y^\mu(x, [\varphi])(\partial/\partial\varphi^\mu)$ consider the maps $I_Y : E_1^{p,m} \to E_1^{p-1,m}$ and $\mathcal{L}_Y : E_1^{p,m} \to E_1^{p,m}$,

$$I_Y[\omega] := [\iota_{\mathcal{P}_Y}\omega], \tag{3.16}$$

$$\mathcal{L}_Y[\omega] := [\mathcal{L}_{\mathcal{P}_Y}\omega], \tag{3.17}$$

where $\mathcal{P}_Y \in \mathrm{Vect}(J)$ is defined by (2.31). These maps are well-defined, because $\iota_{\mathcal{P}_Y}$ and $\mathcal{L}_{\mathcal{P}_Y}$ commute with D (see (2.37) and (2.39)), so for an arbitrary form $\tau \in \Omega^{p,m-1}$, $\iota_{\mathcal{P}_Y}\mathrm{D}\tau = -\mathrm{D}(\iota_{\mathcal{P}_Y}\tau)$ and $\mathcal{L}_{\mathcal{P}_Y}\mathrm{D}\tau = \mathrm{D}(\mathcal{L}_{\mathcal{P}_Y}\tau)$. From (2.38) it also follows that

$$[\mathrm{d}_1, I_Y] = \mathcal{L}_Y \tag{3.18}$$

on classes of forms in $E_1^{p,m}$.

In particular, let $[\omega] = [\Gamma_\sigma^\mu B_\mu^\sigma \mathcal{D}x]$ be an arbitrary class in $E_1^{1,m}$ and $\omega' = \rho(\omega) = \Gamma^\mu \tilde{B}_\mu \mathcal{D}x$ be its canonical representative ($\tilde{B}_\mu = (-1)^{|\sigma|}\mathrm{D}_\sigma^{|s|} B_\mu^\sigma$). Then it follows from (3.16) that for an arbitrary evolutionary vector field Y

$$I_Y[\omega] = I_Y[\omega'] = [Y^\mu \tilde{B}_\mu \mathcal{D}x]. \tag{3.19}$$

Consider a Lagrangian $\mathbf{L} = L\mathcal{D}x$. Let $Y = Y^\mu(x, [\varphi])(\partial/\partial\varphi^\mu)$ be an arbitrary evolutionary vector field. For a section $s(x) = (x, \varphi(x))$ of the bundle E, the value of Y on s gives an infinitesimal variation:

$$\varphi^\mu(x) \mapsto \varphi_\varepsilon^\mu(x) = \varphi^\mu(x) + \varepsilon Y^\mu(x, [\varphi(x)]). \tag{3.20}$$

Then

$$\mathbf{L}|_s \mapsto \mathbf{L}|_{s_\varepsilon} = \mathbf{L}|_s + \varepsilon(\mathcal{L}_{\mathcal{P}_Y}\mathbf{L})|_s, \tag{3.21}$$

and

$$S[\varphi] \mapsto S[\varphi_\varepsilon] = S[\varphi] + \varepsilon \int (\mathcal{L}_{\mathcal{P}_Y}\mathbf{L})|_s. \tag{3.22}$$

From the results above we know that $\mathcal{L}_{\mathcal{P}_Y}(\mathbf{L}) = \iota_{\mathcal{P}_Y}\delta\mathbf{L} + \delta\iota_{\mathcal{P}_Y}\mathbf{L} = \iota_{\mathcal{P}_Y}\delta\mathbf{L}$. It follows from (3.15) and (3.19) that

$$[\mathcal{L}_{\mathcal{P}_Y}(\mathbf{L})] = [\iota_{\mathcal{P}_Y}\delta\mathbf{L}] = I_Y[\delta\mathbf{L}] = [Y^\mu \mathcal{F}_\mu(\mathbf{L})\mathcal{D}x]. \tag{3.23}$$

Since the integral is constant on classes (for compactly supported forms), we conclude that

$$S[\varphi_\varepsilon] = S[\varphi] + \varepsilon \int (Y^\mu \mathcal{F}_\mu(\mathbf{L}))|_s \mathcal{D}x. \tag{3.24}$$

In other words, these considerations make it possible to recover in a purely algebraic way the Euler–Lagrange formula for the variation of action induced by a variation of section.

To summarize: working with classes in $E_1^{p,m}$ is an algebraic model of integration; passing to other representative inside a class, e.g. getting Γ^μ's from Γ^μ_σ's as above, corresponds to classical integration by parts. With this in mind, we rewrite the variational complex (3.7) as

$$0 \to \Omega^0_{\text{hor}} \xrightarrow{\text{D}} \Omega^1_{\text{hor}} \xrightarrow{\text{D}} \cdots \xrightarrow{\text{D}} \Omega^m_{\text{hor}} \xrightarrow{\delta} \int \Omega^{1,m} \xrightarrow{\delta} \int \Omega^{2,m} \xrightarrow{\delta} \int \Omega^{3,m} \xrightarrow{\delta} \cdots \tag{3.25}$$

with a suggestive notation $\int \Omega^{p,m} := E_1^{p,m} = \Omega^{p,m}/\text{D}(\Omega^{p,m-1})$ and restoring δ for d_1, as acting on classes. The formula (3.15) is rewritten then as

$$\delta[\mathbf{L}] = [\delta\varphi^\mu \mathcal{F}_\mu(\mathbf{L})\mathcal{D}x], \tag{3.26}$$

which is identical (even to the letter δ) with the classical formula for the variation of the functional S (which we can identify with $[\mathbf{L}]$). The Cartan forms $\Gamma^\mu = \delta\varphi^\mu$ play the role of "abstract variations", similar to differentials $\text{d}x^a$ in usual calculus, while evolutionary vector fields correspond to actual variations of sections, analogous to vectors in usual calculus. The formula (3.24) corresponds to "taking value" of the "differential" (3.26) on a "vector" Y.

4. The complex of variational derivatives and relations of two complexes

4.1. The complex of variational derivatives

In this section we shall consider another complex related with Euler–Lagrange equations: the complex of Lagrangians of parameterized surfaces in a given manifold M. Actually, we shall perform all considerations for an arbitrary supermanifold. The variational complex considered above can be generalized for supercase, too (see Section 4.2).

Let $M = M^{m|n}$ be a supermanifold with coordinates x^a. Consider an $r|s$-dimensional coordinate superspace $\mathbb{R}^{r|s}$ with standard coordinates t^i. In our notation some of coordinates are even, some odd. The parity of indices is the parity of the respective coordinates, $\tilde{a} := \tilde{x}^a$, etc. In the sequel we shall often omit the prefix "super" (unless required for clarity). Consider smooth maps of a neighborhood of zero in $\mathbb{R}^{r|s}$ to M. Consider the supermanifold of jets of such maps, of order k, at point zero. (In the supercase jets are defined exactly as in the purely even case.) Denote it by $T^{(k)}_{r|s}M$. In local coordinates the elements of $T^{(k)}_{r|s}M$ will be $[x] = (x^a_\sigma) = (x^a, x^a_i, \ldots, x^a_{i_1\cdots i_k})$. There are natural projections $T^{(k+1)}_{r|s}M \to T^{(k)}_{r|s}M$. Consider the inverse limit $T^\infty_{r|s}M$ of the sequence of bundles:

$$\cdots \to T^{(k+1)}_{r|s}M \to T^{(k)}_{r|s}M \cdots \to T^{(1)}_{r|s}M \to T^{(0)}_{r|s}M = M. \tag{4.1}$$

Definition 4.1. We call the manifold $T^{(k)}_{r|s}M$ the manifold of *tangent* $r|s$*-elements of order* k ($k = 0, 1, 2, \dots, \infty$).

Tangent elements of infinite order will be shortly called *tangent elements*. Clearly, tangent $1|0$-elements of the first order are simply (even) tangent vectors: $T^{(1)}_{1|0}M = TM$. Tangent $r|s$-elements of the first order are arrays of r even and s odd tangent vectors. Notice that for $k > 1$, the bundle $T^{(k)}_{r|s}M \to M$ is not a vector bundle.

We define $r|s$*-Lagrangians* on M as smooth functions on the space of $r|s$-tangent elements $T^{\infty}_{r|s}M$. (As before, we consider functions of finite order.) Denote the space of all $r|s$*-Lagrangians* by $\Phi^{r|s} = \Phi^{r|s}(M)$.

Consider $r|s$*-paths* ($r|s$-dimensional parameterized surfaces), i.e. maps $\gamma : U^{r|s} \to M$ (where $U^{r|s} \subset \mathbb{R}^{r|s}$). Notice that there is no condition like immersion at this moment. Every $r|s$-path $x^a = x^a(t)$ at the point t defines the tangent element

$$[x(t)] = \left(x^a(t), \frac{\partial x^a}{\partial t}(t), \frac{\partial^2 x^a}{\partial t^i \partial t^j}(t), \dots \right). \tag{4.2}$$

Every Lagrangian $L \in \Phi^{r|s}(M)$ defines a functional on $r|s$*-paths*:

$$S[\gamma] = \int_{U^{r|s}} L([x(t)])\mathcal{D}t, \tag{4.3}$$

where $\mathcal{D}t$ is the standard coordinate volume form on $U^{r|s} \subset \mathbb{R}^{r|s}$.

Remark 4.1. There are two essential differences with the setup of the previous section. First, the space of parameters $\mathbb{R}^{r|s}$ is endowed with fixed coordinates t^i. This allows to consider the “standard” volume element $\mathcal{D}t$ and to define our Lagrangians as functions (not as forms on jet space). Second, Lagrangians considered here are geometrical objects on M (not $\mathbb{R}^{r|s}$ or $\mathbb{R}^{r|s} \times M$), hence cannot depend on t^i.

From the variational problem for the functional (4.3) one can easily obtain the following formula for the variational derivative:

$$\mathcal{F}_a(L) = \frac{\partial L}{\partial x^a} - (-1)^{\tilde{a}\tilde{\imath}}\, \mathrm{D}_i\left(\frac{\partial L}{\partial x^a_i}\right) + \cdots = \sum_{|\sigma|=0}^{\infty} (-1)^{|\sigma| + \tilde{a}\tilde{\sigma}}\, \mathrm{D}^{|\sigma|}_{\sigma}\left(\frac{\partial L}{\partial x^a_i}\right). \tag{4.4}$$

Here D_i stands for the total derivative with respect to t^i (covariant derivative in the terminology of the previous sections):

$$\mathrm{D}_i = x^a_i \frac{\partial}{\partial x^a} + x^a_{ij}\frac{\partial}{\partial x^a_j} + \cdots. \tag{4.5}$$

(Clearly, $\mathcal{F}_a(L) = \mathcal{F}_a(\mathbf{L})$ for the Lagrangian $\mathbf{L} = L\mathcal{D}t$ considered as a form, compare with the next subsection on the variational complex in the supercase.)

Definition 4.2. The *differential* of an $r|s$-Lagrangian L is the $(r+1|s)$-Lagrangian $đL$ defined by the formula [16]

$$đL = x^a_{r+1}\mathcal{F}_a(L). \tag{4.6}$$

Here $\bar{d}L$ depends on $x^a_{i_1\cdots i_k}$ where the lower indices run over the set that includes one more even index $r+1$, corresponding to the new even variable t^{r+1}. We call the operator $\bar{d}$ the *variational differential*.

Proposition 4.1 (see [16]). $\bar{d}^2 = 0$.

We arrive at the *complex of variational derivatives* on a (super)manifold M:

$$0 \to \Phi^{0|s} \xrightarrow{\bar{d}} \Phi^{1|s} \xrightarrow{\bar{d}} \Phi^{2|s} \xrightarrow{\bar{d}} \cdots . \tag{4.7}$$

Remark 4.2. In the purely even case, an example of r-Lagrangian of the first order is provided by an arbitrary r-form. It is easy to see that in this case the operator $\bar{d}$ gives the usual Cartan–de Rham exterior differential on forms. To express the exterior differential in terms of variational derivatives was the original idea of one of the authors dating back to [15]. The motivation was to get the correct definition of the Cartan–de Rham complex in the supercase (see [15,18,19]), see Section 4.3. On the other hand, for general Lagrangians, it seems geometrically very natural to have a complex where "variation of action" directly gives the differential. This is achieved by the identification of the number of independent variables with the cohomological degree increased by the action of the differential.

An application of the complex (4.7) to symmetries of Lagrangians was given in [5].

4.2. Variational complex in super context

Much of the considerations of Sections 2 and 3 carries over to supermanifolds without difficulties. A tricky thing is the choice of the precise class of forms that should be used. The analog of the Cartan–de Rham complex for supermanifolds in its full generality is highly nontrivial (see Remarks 4.3 and 4.4 below). (This was one of the sources of the current research, in particular, of the study of the complex of variational derivatives, see Section 4.4.) However, for the purposes of this paper, these complications can be circumvented, as it is shown below, if we are only interested in forms necessary to construct the super version of the variational complex (3.25).

Let $E \to M$ be a fibre bundle where both E and M are supermanifolds. Let $\dim M = m|n$ (dimension of the base). We continue to use the coordinates x^a for M and φ^μ for the fibre. Some of them are even, some odd. The parity for all objects is denoted by tilde. The jet bundles, in particular, the infinite jet bundle $J \to M$, are defined exactly as in the purely even case. The natural coordinates for jets, φ^μ_σ, have parity of the corresponding partial derivatives: $\tilde{\varphi}^\mu_\sigma = \tilde{\mu} + \tilde{\sigma}$, where $\tilde{\sigma} := \tilde{a}_1 + \cdots + \tilde{a}_k$ for a multi-index $\sigma = a_1 \cdots a_k$. They enjoy the symmetry property $\varphi^\mu_{\sigma ab\tau} = (-1)^{\tilde{a}\tilde{b}} \varphi^\mu_{\sigma ba\tau}$, for each pair of neighbor indices. As before, all functions are of finite order, i.e. depend on a finite number of the coordinates φ^μ_σ. Vector fields can have infinitely many nonzero coefficients.

Let ΠTJ be the antitangent bundle for J, i.e. the tangent bundle with reversed parity of fibres. The natural local coordinates in ΠTJ are x^a, φ^μ_σ, dx^a, $d\varphi^\mu_\sigma$. The differentials are treated as independent variables of parity opposite to that of the respective coordinates. We shall consider various functions on ΠTJ including generalized functions w.r.t. the variables

$\mathrm{d}x^a$ and $\mathrm{d}\varphi^\mu_\sigma$. The space of all such functions (without specifying a class) will be denoted here by $\Omega(J)$. Warning: it is not an algebra, as not for all functions multiplication is defined. In particular, inside $\Omega(J)$ is contained the algebra of "naïve" differential forms, consisting of functions polynomial in differentials. The exterior differential d can be viewed as an odd vector field on the supermanifold ΠTJ, $\mathrm{d}(x^a) = \mathrm{d}x^a$, $\mathrm{d}(\varphi^\mu_\sigma) = \mathrm{d}\varphi^\mu_\sigma$, $\mathrm{d}(\mathrm{d}x^a) = 0$, $\mathrm{d}(\mathrm{d}\varphi^\mu_\sigma) = 0$.

Remark 4.3. On arbitrary supermanifold, the straightforward analog of usual differential forms (free supercommutative algebra generated by differentials of coordinates) does not contain objects of integration like volume forms, because the "super-Jacobian" (the Berezinian) is a fraction, not a polynomial. So such "naïve" forms make only a part of quite tricky picture of forms on supermanifolds, see, in particular, [18,19]. In the context of this paper it is possible to simplify our task by considering, as we do, a particular class of the so-called Bernstein–Leites pseudodifferential forms [2,18,19] instead of arbitrary super forms.

The Cartan connection form is defined as before,

$$\Gamma = \Gamma^\mu_\sigma \frac{\partial}{\partial \varphi^\mu_\sigma}, \tag{4.8}$$

where

$$\Gamma^\mu_\sigma = \mathrm{d}\varphi^\mu_\sigma - \mathrm{d}x^a \varphi^\mu_{a\sigma}. \tag{4.9}$$

It defines the decomposition of tangent and cotangent spaces and their opposites (spaces with reversed parity) into the horizontal and vertical subspaces. In particular, for the tangent space a basis of the horizontal subspace consists of partial covariant derivatives

$$\mathrm{D}_a = \frac{\partial}{\partial x^a} + \varphi^\mu_{a\sigma} \frac{\partial}{\partial \varphi^\mu_\sigma}, \tag{4.10}$$

and a basis of the vertical subspace consists of the derivatives $\partial/\partial\varphi^\mu_\sigma$. For the anticotangent space, a basis of the horizontal subspace consists of the differentials $\mathrm{d}x^a$ and a basis of the vertical subspace consists of the forms Γ^μ_σ. These bases of vectors and 1-forms are dual. It is convenient to use $\mathrm{d}x^a$ and Γ^μ_σ as fibre coordinates in ΠTJ instead of $\mathrm{d}x^a$ and $\mathrm{d}\varphi^\mu_\sigma$. Notice that

$$\mathrm{d}\Gamma^\mu_\sigma = (-1)^{\tilde{a}}\, \mathrm{d}x^a \Gamma^\mu_{a\sigma}. \tag{4.11}$$

The form-valued vector field Γ gives the Lie derivative $\mathcal{L}_\Gamma$, which is a vector field on ΠTJ generating the infinitesimal transformation

$$\varphi^\mu_\sigma \mapsto \varphi^\mu_\sigma + \varepsilon \Gamma^\mu_\sigma, \tag{4.12}$$

$$\mathrm{d}\varphi^\mu_\sigma \mapsto \mathrm{d}\varphi^\mu_\sigma - \varepsilon\, \mathrm{d}\Gamma^\mu_\sigma = \mathrm{d}\varphi^\mu_\sigma - \varepsilon(-1)^{\tilde{a}} \Gamma^\mu_{a\sigma} \tag{4.13}$$

(here ε is an odd constant, $\varepsilon^2 = 0$). It follows that $\Gamma^\mu_\sigma \mapsto \Gamma^\mu_\sigma$, i.e.

$$\mathcal{L}_\Gamma \Gamma^\mu_\sigma = 0, \tag{4.14}$$

which is equivalent to the vanishing of the Nijenhuis bracket

$$[\Gamma, \Gamma] = 0 \tag{4.15}$$

(flatness of the connection Γ). Hence, we can decompose $\mathrm{d} = \mathrm{D} + \delta$: if $\omega = \omega(x, [\varphi], \mathrm{d}x, \Gamma)$ is a function on ΠTJ written in the coordinates x^a, φ^μ_σ, $\mathrm{d}x^a$, Γ^μ_σ, then

$$\mathrm{d}\omega = \underbrace{\mathrm{d}x^a \left(\frac{\partial \omega}{\partial x^a} + \varphi^\mu_{a\sigma} \frac{\partial \omega}{\partial \varphi^\mu_\sigma} + (-1)^{\tilde{a}} \Gamma^\mu_{a\sigma} \frac{\partial \omega}{\partial \Gamma^\mu_\sigma} \right)}_{\text{horizontal}} + \underbrace{\Gamma^\mu_\sigma \frac{\partial \omega}{\partial \varphi^\mu_\sigma}}_{\text{vertical}}, \tag{4.16}$$

or

$$\mathrm{D} = \mathrm{d}x^a \left(\frac{\partial}{\partial x^a} + \varphi^\mu_{a\sigma} \frac{\partial}{\partial \varphi^\mu_\sigma} + (-1)^{\tilde{a}} \Gamma^\mu_{a\sigma} \frac{\partial}{\partial \Gamma^\mu_\sigma} \right), \tag{4.17}$$

$$\delta = \mathcal{L}_\Gamma = \Gamma^\mu_\sigma \frac{\partial}{\partial \varphi^\mu_\sigma}. \tag{4.18}$$

The differentials D and δ commute: $\mathrm{D}\delta + \delta\,\mathrm{D} = 0$, and $\mathrm{D}^2 = \delta^2 = 0$. In particular,

$$\Gamma^\mu_\sigma = \delta \varphi^\mu_\sigma. \tag{4.19}$$

One can introduce a bigrading into the algebra of naïve differential forms exactly as in the purely even case considered in Section 2:

$$\Omega^{p,q}(J) := \{\omega \in \Omega(J) | \#\Gamma^\mu_\sigma(\omega) = p \ \text{ and } \ \#\mathrm{d}x^a(\omega) = q\}, \tag{4.20}$$

where ω is assumed to be polynomial in $\mathrm{d}x^a$ and Γ^μ_σ, and $\#\mathrm{d}x^a$, $\#\Gamma^\mu_\sigma$ stand for the degree in the corresponding variables. It is possible to carry on with the filtration and spectral sequence as above. The trouble is that in the super case these considerations give nothing for the variational problem: since naïve differential forms cannot be integrated over supermanifolds (see Remark 4.3), Lagrangians are not contained in the complex $\Omega^{**}(J)$.

The correct class of forms on J, adequate for our purposes, can be introduced as follows. (This new class of forms particularly designed for the needs of variational complex may be interpreted as hybrids of “integral forms” and naïve differential forms, see Remark 4.4.)

Recall that for arbitrary “super” variables z^a the delta-function $\delta(z)$ corresponds to the distribution $f \mapsto f(0)$ w.r.t. the Berezin integration in variables z^a. In particular, for even variables it is the usual delta-function, and for odd variables ξ^i

$$\delta(\xi^1, \ldots, \xi^n) = \xi^n \xi^{n-1} \cdots \xi^1 \tag{4.21}$$

(maximal product; notice the inverse order). Delta-function $\delta(z)$ satisfies the property

$$\delta(z) = \delta(z') \cdot \left(\mathrm{Ber} \frac{\partial z}{\partial z'} \right)^{-1} \mathrm{sign}\,\det \left(\frac{\partial z}{\partial z'} \right)_{00} \tag{4.22}$$

for a non-singular change of variables, where Ber is the Berezinian and the indices 00 refer to the even–even block of the Jacobi matrix. Consider the delta-function $\delta(\mathrm{d}x)$ of the variables $\mathrm{d}x^a$ (no confusion should be with the vertical differential δ!). This is well-defined, because the differentials $\mathrm{d}x^a$ transform through themselves under changes of variables x^a

and φ^μ_σ. Moreover, from the property (4.22) we obtain the following transformation law:

$$\delta(\mathrm{d}x) = \delta(\mathrm{d}x')\cdot\left(\mathrm{Ber}\left(\frac{\partial x}{\partial x'}\right)^{\Pi}\right)^{-1}\mathrm{sign}\det\left(\frac{\partial x}{\partial x'}\right)_{11}$$
$$= \delta(\mathrm{d}x')\cdot\mathrm{Ber}\frac{\partial x}{\partial x'}\cdot\mathrm{sign}\det\left(\frac{\partial x}{\partial x'}\right)_{11}, \tag{4.23}$$

where the superscript Π denotes reversion of parity of rows and columns of a matrix (notice that $\mathrm{Ber}\, g^{\Pi} = (\mathrm{Ber}\, g)^{-1}$). It follows that up to a sign factor, $\delta(\mathrm{d}x)$ transforms exactly as the Berezin volume element $\mathcal{D}x$ and thus can be identified with such. More precisely, we consider (generalized) functions on ΠTJ that are supported at the closed submanifold $\Pi VJ \hookrightarrow \Pi TJ$ (VJ is the vertical subbundle), which is locally specified by the equations $\mathrm{d}x^a = 0$. Let them take values in the local system $\mathrm{sign}\det(TM)_1$. This eliminates the above sign factor. Now, every such function is a linear combination of the delta-function $\delta(\mathrm{d}x)$ and its derivatives. We introduce the following notation:

$$\mathcal{D}x := \delta(\mathrm{d}x), \tag{4.24}$$

$$\mathcal{D}x_{a_1\cdots a_k} := \frac{\partial}{\partial\, \mathrm{d}x^{a_1}}\cdots\frac{\partial}{\partial\, \mathrm{d}x^{a_k}}\delta(\mathrm{d}x). \tag{4.25}$$

After introducing the said local coefficients, the formula (4.24) is a genuine identification. (In particular, for a purely even manifold M^m, we have $\mathcal{D}x = \delta(\mathrm{d}x) = \mathrm{d}x^m\,\mathrm{d}x^{m-1}\cdots\mathrm{d}x^1 = \pm\mathrm{d}x^1\cdots\mathrm{d}x^m$; notice slight difference of sign from our notation in Section 3.) The forms that we need have the appearance:

$$\omega = \frac{1}{k!}\omega^{a_1\cdots a_k}\mathcal{D}x_{a_k\cdots a_1} \tag{4.26}$$

($k!$ and the order of indices are chosen for convenience). The coefficients $\omega^{a_1\cdots a_k}$ depend on x^a, φ^μ_σ and Γ^μ_σ. From (4.24) and (4.25) follow the rules of operations with the symbols $\mathcal{D}x$ and $\mathcal{D}x_{a_1\cdots a_k}$:

$$\mathrm{d}x^a\,\mathcal{D}x = 0, \tag{4.27}$$

$$\frac{\partial}{\partial\, \mathrm{d}x^a}\mathcal{D}x_{a_1\cdots a_k} = \mathcal{D}x_{aa_1\cdots a_k}, \tag{4.28}$$

$$\mathrm{d}x^a\,\mathcal{D}x_{a_1\cdots a_k} = \sum_{i=1}^{k}(-1)^{\tilde a+(\tilde a+1)(\alpha_1+\cdots+\tilde a_{i-1}+i-1)}\delta^a_{a_i}\mathcal{D}x_{a_1\cdots\hat a_i\cdots a_k}, \tag{4.29}$$

where hat means that the index is dropped. In particular, for arbitrary a,

$$\underbrace{\mathrm{d}x^a\mathcal{D}x_a}_{\text{no summation}} = (-1)^{\tilde a}\mathcal{D}x. \tag{4.30}$$

The space of forms (4.26) is not an algebra, but as follows from (4.27)–(4.29), it is a module over the “naïve” algebra of polynomial forms, of “Fock type”, where multiplication by $\mathrm{d}x^a$ acts as annihilation operators for a “vacuum vector” $\mathcal{D}x$.

Consider forms (4.26) with coefficients polynomial in Γ_σ^μ. Define

$$\Sigma^{p,q}(J) := \left\{ \omega = \frac{1}{k!}\omega^{a_1\cdots a_k}\mathcal{D}x_{a_k\cdots a_1} | \#\Gamma_\sigma^\mu(\omega) = p \text{ and } k = m - q \right\}. \tag{4.31}$$

(We have $\Sigma^{p,q}(J) \subset \Omega(J) \otimes \mathcal{E}$, where $\mathcal{E}$ is the orienting sheaf for $(TM)_1$.) Here m is the even dimension of the base: $M = M^{m|n}$. Operations D and δ act on Σ^{**}. For ω as in (4.26), one can find from (4.17), (4.18), (4.27)–(4.29) that D acts as a "divergence":

$$\mathrm{D}\omega = -\frac{1}{(k-1)!}(-1)^{(\tilde{\omega}+m)(\tilde{a}+1)}\,\mathrm{D}_a\omega^{ab_1\cdots b_{k-1}}\mathcal{D}x_{b_{k-1}\cdots b_1}, \tag{4.32}$$

and δ acts componentwise:

$$\delta\omega = \frac{1}{k!}\Gamma_\sigma^\mu \frac{\partial\omega^{a_1\cdots a_k}}{\partial\varphi_\sigma^\mu}\mathcal{D}x_{a_k\cdots a_1}. \tag{4.33}$$

Notice that $\mathrm{D} : \Sigma^{p,q} \to \Sigma^{p,q+1}$ and $\delta : \Sigma^{p,q} \to \Sigma^{p+1,q}$. The bicomplex Σ^{**} is bounded at the bottom and at the right: $\Sigma^{p,q} = 0$ for $p < 0$ or $q > m$. In particular, there is a complex of horizontal forms

$$\cdots \to \Sigma^0_{\mathrm{hor}} \xrightarrow{\mathrm{D}} \Sigma^1_{\mathrm{hor}} \xrightarrow{\mathrm{D}} \cdots \xrightarrow{\mathrm{D}} \Sigma^{m-1}_{\mathrm{hor}} \xrightarrow{\mathrm{D}} \Sigma^m_{\mathrm{hor}} \to 0, \tag{4.34}$$

where $\Sigma^q_{\mathrm{hor}} = \Sigma^{0,q}$. (It is not bounded at the left.)

Applying the machinery of homological algebra to the bicomplex $\Sigma^{**}(J)$, we arrive at a spectral sequence analogous to (3.2). In particular, we obtain the *variational complex* for the super case:

$$\cdots \to \Sigma^0_{\mathrm{hor}} \xrightarrow{\mathrm{D}} \Sigma^1_{\mathrm{hor}} \xrightarrow{\mathrm{D}} \cdots \xrightarrow{\mathrm{D}} \Sigma^m_{\mathrm{hor}} \xrightarrow{\delta} \int \Sigma^{1,m} \xrightarrow{\delta} \int \Sigma^{2,m} \xrightarrow{\delta} \int \Sigma^{3,m} \xrightarrow{\delta} \cdots. \tag{4.35}$$

Here the elements of the space $\int \Sigma^{p,m}$ are formal integrals $\int \omega$, where $\omega \in \Sigma^{p,m}$. These formal integrals are in 1–1-correspondence with classes ω mod $\mathrm{D}(\Sigma^{p,m-1})$. One subtlety is that for consistence with the parity of the genuine Berezin integral we define the isomorphism $\int \Sigma^{p,m} \cong E_1^{p,m} = \Sigma^{p,m}/\mathrm{D}(\Sigma^{p,m-1})$ as having parity n (where n is the odd dimension of the base $M = M^{m|n}$). Notice the following properties of this complex: it is not bounded from the left (differently from (3.25)) and it is not bounded from the right (similar to (3.25)). Still, its cohomology is the ordinary cohomology of the underlying topological space of the bundle E.

The relation with the classical calculus of variations is exactly as in the purely even case. Suppose we have a Lagrangian $\mathbf{L} = L(x, [\varphi])\mathcal{D}x$. It is an element of $\Sigma^{0,m}$. The action (treated formally) is the class $\int \mathbf{L} \in \int \Sigma^{0,m} \cong \Sigma^{0,m}/\mathrm{D}(\Sigma^{0,m-1})$. Consider $\delta \int \mathbf{L}$ (since δ and D commute, it makes sense to apply δ modulo D-coboundaries). We have,

$$\delta \int \mathbf{L} = (-1)^n \int \delta\mathbf{L} = (-1)^n \int \Gamma_\sigma^\mu \frac{\partial L}{\partial\varphi_\sigma^\mu}\mathcal{D}x. \tag{4.36}$$

To transform this, consider first an arbitrary form $\omega \in \Sigma^{1,m}$. Let

$$\omega = \Gamma^\mu_{a_1\cdots a_k} B_\mu^{a_1\cdots a_k}\mathcal{D}x = \delta\varphi^\mu_{a_1\cdots a_k} B_\mu^{a_1\cdots a_k}\mathcal{D}x. \tag{4.37}$$

Exactly as we did to obtain (3.13), we can show, using (4.30), that there is an equality modulo $\mathrm{D}(\Sigma^{1,m-1})$:

$$\omega = \delta\varphi^{\mu}_{a_1\cdots a_k} B^{a_1\cdots a_k}_{\mu}\mathcal{D}x = -(-1)^{\tilde{a}_k\tilde{\mu}}\delta\varphi^{\mu}_{a_1\cdots a_{k-1}}\,\mathrm{D}_{a_k} B^{a_1\cdots a_k}_{\mu}\mathcal{D}x.$$

From here we deduce the map ρ giving canonical representatives: $\omega \equiv \rho(\omega) \bmod \mathrm{D}(\Sigma^{1,m-1})$, where

$$\rho(\omega) = (-1)^{|\sigma|+\tilde{\sigma}\tilde{\mu}}\delta\varphi^{\mu}\,\mathrm{D}^{|\sigma|}_{\sigma} B^{\sigma}_{\mu}\mathcal{D}x, \tag{4.38}$$

where $\sigma = a_1\cdots a_k$. Applying this to (4.36), we obtain

$$\delta\int \mathbf{L} = (-1)^n \int \delta\varphi^{\mu}(-1)^{|\sigma|+\tilde{\sigma}\tilde{\mu}}\,\mathrm{D}^{|\sigma|}_{\sigma}\frac{\partial L}{\partial\varphi^{\mu}_{\sigma}}\mathcal{D}x, \tag{4.39}$$

which is exactly

$$\delta\int \mathbf{L} = (-1)^n \int \delta\varphi^{\mu}\mathcal{F}_{\mu}(\mathbf{L})\mathcal{D}x, \tag{4.40}$$

with

$$\mathcal{F}_{\mu}(\mathbf{L}) = (-1)^{|\sigma|+\tilde{\sigma}\tilde{\mu}}\,\mathrm{D}^{|\sigma|}_{\sigma}\frac{\partial L}{\partial\varphi^{\mu}_{\sigma}} \tag{4.41}$$

being the variational derivative in the super case.

Remark 4.4. Forms that we introduce here are hybrids of Bernstein–Leites integral forms (multivector densities of weight 1) on the base M with differential forms on fibres. This can be seen directly if one applies the Fourier–Hodge transform to our formulas (4.24)–(4.26). We preferred not to work with a “hybrid” definition explicitly in order to avoid a nonlocal transformation law under coordinate changes (though it would not occur if one sticks to the non-holonomic frame Γ^{μ}_{σ}), choosing instead the language of generalized functions. To put this into a proper framework, one should notice that, in general, the space of forms on a supermanifold is $\mathbf{\Omega}^{r|s}$ (see [19]), where $r|s$ is a super dimension. For the jet space J the most general bicomplex would carry a bigrading like $\mathbf{\Omega}^{r|s,p|q}$. The naïve spaces of forms $\Omega^{p,q}$ that we initially introduced are exactly $\mathbf{\Omega}^{p|0,q|0}$; the “correct” space $\Sigma^{p,q}$ that we used for the variational complex coincides with $\mathbf{\Omega}^{p|0,q|n}$ (here n is the odd dimension of the base $M^{m|n}$).

Remark 4.5. The calculus of variations for the super case was considered for the first time by Martin, in a pioneering work [9]. In particular, the idea of an integral over odd variables as a class modulo total derivatives dates back to that paper. Among recent works, the paper [13] should be mentioned. It treats Lagrangian formalism in an algebraic setting of quite general graded-commutative algebras. There is little overlapping with our present analysis, however, because in [13] the emphasis is on defining such objects as the Berezinian, the Berezin integral, etc. in an algebraic fashion and in great generality, and here we start from integration theory for supermanifolds considered known (see, e.g. [18,19]) and develop our theory upon it—the crucial thing is singling out the proper classes of forms.

4.3. Relation of complexes

Let us compare the variational complex (3.7) and (3.25) with the complex of variational derivatives (4.7). For simplicity consider a purely even case. (The general super case is similar.) Consider for every r the trivial fiber bundle $\pi_{(r)} = \mathbb{R}^r \times M$ with base $\mathbb{R}^r$ and fibre M and the corresponding space of jets $J^\infty(\pi_{(r)})$. In the sequel we shall shortly denote $J_{(r)} := J^\infty(\pi_{(r)})$. There is a natural projection $J_{(r)} \to T_r^{(\infty)}$ onto the bundle of tangent elements of M.

Consider the embedding $\Omega^*_{(r)} \hookrightarrow \Omega^*_{(r+1)}$ induced by the natural projection $p_r : \pi_{(r+1)} \to \pi_{(r)}$, where $\Omega^*_{(r)}$ is the space of differential forms on the space $J_{(r)}$. Consider the filtrations (3.1) generated by the Cartan ideals in the spaces $\Omega^*_{(r)}$ and $\Omega^*_{(r+1)}$ and the zero terms $E^{**}_{(r)0}$, $E^*_{(r+1)0}$ of the corresponding spectral sequences (see (3.3)).

Proposition 4.2.

1. *Under the projection p_r, the kth power of the Cartan ideal in $\Omega^*_{(r)}$ maps to the $(k-1)$-th power of the Cartan ideal in the space $\Omega^*_{(r+1)}$:*

$$p_r^* : C^k\Omega^*_{(r)} \to C^{k-1}\Omega^*_{(r+1)}. \tag{4.42}$$

2. *The map (4.42) induces a map of the zeroth terms of the corresponding spectral sequences $E^{p,q}_{(r)0} \to E^{p-1,q+1}_{(r+1)0}$. Thus it defines a map of bicomplexes $\Omega^{**}_{(r)} \to \Omega^{**}_{(r+1)}$ of bidegree $(-1, 1)$:*

$$\kappa_r : \Omega^{p,q}_{(r)} \to \Omega^{p-1,q+1}_{(r+1)}. \tag{4.43}$$

Proof. Let $\omega = \omega(x, \mathrm{d}t, \Gamma)$ be a form in $\Omega^{p,q}(J_{(r)})$. Here $\Gamma = \Gamma_{(r)}$ stands for Cartan forms on $J_{(r)}$. Notice that $\Gamma^a_{(r)\sigma} = \Gamma^a_{(r+1)\sigma} + \mathrm{d}t^{r+1}x^a_{r+1,\sigma}$. Hence,

$$p_r^*\omega = \omega(x, \mathrm{d}t, \Gamma_{(r)}) = \omega(x, \mathrm{d}t, \Gamma_{(r+1)}) + \mathrm{d}t^{r+1}x^a_{r+1,\sigma}\frac{\partial\omega}{\partial\Gamma^a_{(r)\sigma}}. \tag{4.44}$$

The first term in the RHS belongs to $\Omega^{p,q}(J_{(r+1)})$ and the second term in the RHS belongs to $\Omega^{p-1,q+1}(J_{(r+1)})$. It follows that the induced map $\kappa_r : \Omega^{p,q}(J_{(r)}) \to \Omega^{p-1,q+1}(J_{(r+1)})$ is given by the formula

$$\kappa_r\omega = \mathrm{d}t^{r+1}x^a_{r+1,\sigma}\frac{\partial\omega}{\partial\Gamma^a_{(r)\sigma}}. \qquad \square \tag{4.45}$$

For example, let $\omega = \Gamma^a_{(r)}\Gamma^b_{(r)i} \in \Omega^{2,0}_{(r)}$ be a differential form on $J_{(r)}$, where

$$\Gamma^a_{(r)} = \mathrm{d}x^a - \sum_{i\le r}\mathrm{d}t^i x^a_i, \qquad \Gamma^b_{(r)i} = \mathrm{d}x^b_i - \sum_{j\le r}\mathrm{d}t^j x^b_{ij}. \tag{4.46}$$

Under the embedding (4.42) the form $\omega = \Gamma^a_{(r)}\Gamma^b_{(r)i}$ gives the form

$$p_r^*\omega = (\Gamma^a_{(r+1)} + \mathrm{d}t^{r+1}x^a_{r+1})(\Gamma^b_{(r+1)i} + \mathrm{d}t^{r+1}x^b_{r+1,i}) \in C^1\Omega^2_{(r+1)} \tag{4.47}$$

on the space $J_{(r+1)}$. Hence, $\kappa_r \omega = x^a_{r+1} \mathrm{d}t^{r+1} \Gamma^b_{(r+1)i} + \Gamma^a_{(r+1)} x^b_{i.r+1} \, \mathrm{d}t^{r+1}$, which belongs to $\Omega^{1,1}_{(r)}$.

Consider the induced action of κ_r on variational complexes (3.25). Clearly, κ_r vanishes on $\Omega^{0,q}_{(r)}$ and $\kappa_r(\int \Omega^{1,r}_{(r)}) \subset \int \Omega^{0,r+1}_{(r+1)}$. Recall the map ρ that takes classes in $\int \Omega^{1,r}_{(r)}$ to their canonical representatives, see (3.13). Define the map $\chi_r : \int \Omega^{1,r}_{(r+1)} \to \Omega^{0,r+1}_{(r+1)}$ as the composition $\kappa_r \circ \rho$. We shall use the map χ_r to analyze the relation between the variational complex and the complex of variational derivatives (4.7).

Assign to every r-Lagrangian L on M^m the horizontal r-form $\mathbf{L} = L\mathcal{D}t_{(r)}$ on $J_{(r)}$, where $\mathcal{D}t_{(r)}$ is the coordinate volume form on the space of parameters $\mathbb{R}^r$. Consider the $(r+1)$-Lagrangian $\text{đ}L = x^a_{r+1}\mathcal{F}_a(L)$, where $\mathcal{F}_a(L) = \mathcal{F}_a(\mathbf{L})$ is the variational derivative (4.4). To it corresponds the form $\text{đ}L\, \mathcal{D}t_{(r+1)} = x^a_{r+1}\mathcal{F}_a(L)\mathcal{D}t_{(r+1)}$ belonging to $\Omega^{0,r+1}_{(r+1)}$. On the other hand, consider the canonical representative of the element $\delta[\mathbf{L}] \in E^{1,r}_{(r)1}$, which is the form $\rho(\delta[\mathbf{L}]) = \Gamma^a \mathcal{F}_a \mathcal{D}t_{(r)} = (\mathrm{d}x^a - \sum_{1\le i\le r} \mathrm{d}t^i x^a_i)\mathcal{F}_a\mathcal{D}t_{(r)}$ belonging to $\Omega^{1,r}_{(r)}$. Applying κ_r we obtain the action of χ_r on $\delta[L\mathcal{D}t_{(r)}]$. Thus we have the following proposition:

Proposition 4.3. *For every r-Lagrangian L*

$$\text{đ}L\, \mathcal{D}t_{(r+1)} = \chi_r(\delta[L\mathcal{D}t_{(r)}]). \tag{4.48}$$

Remark 4.6. Vanishing of the induced map $\kappa_r : \int \Omega^{1,r}_{(r)} \to \int \Omega^{0,r}_{(r+1)}$ on the image of the map $\delta : \Omega^{0,r}_{(r)} \to \int \Omega^{1,r}_{(r)}$ together with the formula (4.48) implies that $\text{đ}L\, \mathcal{D}t_{(r+1)}$ is a D-coboundary, i.e. $\text{đ}L$ is a total divergence.

4.4. The complex of variational derivatives and covariant Lagrangians

Consider $r|s$-Lagrangians L on a supermanifold M such that for every $r|s$-path $\gamma : x^a = x^a(t)$ the corresponding functional $S[\gamma]$ depends only on the image of the path $x(t)$. That means that if γ_1 and γ_2 are two arbitrary paths such that $x_2(t) = x_1(f(t))$, where f is an orientation preserving diffeomorphism of $\mathbb{R}^{r|s}$, then

$$\int L|_{\gamma_1}\mathcal{D}t = \int L|_{\gamma_2}\mathcal{D}t. \tag{4.49}$$

Definition 4.3. We call L a *covariant $r|s$-Lagrangian of weight* ρ if for arbitrary $r|s$-path γ and an arbitrary orientation preserving diffeomorphism f of $\mathbb{R}^{r|s}$ the following condition holds:

$$L|_{f^*\gamma}(\mathcal{D}t)^\rho = f^*(L|_\gamma(\mathcal{D}t)^\rho). \tag{4.50}$$

If a covariant $r|s$-Lagrangian has weight $\rho = 1$ then this condition is equivalent to the condition (4.49).

Remark 4.7. The condition of covariance (in a more explicit way written below as (4.51)) implies a restriction on allowed paths. Recall that originally nothing like a non-singularity has been imposed. However, because the Berezinian has a denominator, one should require that "odd velocity vectors" in the argument of a covariant Lagrangian of weight $\rho > 0$ be linearly independent. That is, we have to require an immersion in odd directions (no restriction in even directions) for our paths. Such maps were called "proper maps" in [18]. (Similarly, for negative weights, we have to require immersivity in even directions.) In particular, this holds for embedded surfaces. A covariant $r|s$-Lagrangian L of weight ρ on M defines a density $L|_{x(t)}(\mathcal{D}t)^{\rho}$ of weight ρ on an arbitrary $r|s$-dimensional surface embedded in M.

Remark 4.8. Embedded surfaces (submanifolds) from the viewpoint of mathematical physics provide a language where "fields" and "coordinates" are on an equal footing. This approach is due to Schwarz, see, for example, [12]. This language was proved to be powerful in supergravitation. Covariant Lagrangians for embedded surfaces (called "densities" by Schwarz) are important because they provide natural objects of integration over such surfaces. In supermathematics, covariant Lagrangians of the first order (defined *not* only for embedded surfaces, but for arbitrary proper paths) provide a starting point for the definition of forms, see, e.g. [18,19], and below.

In the language of tangent elements the covariance condition (4.50) can be stated as follows. In the space of tangent $r|s$-elements at every point of M acts the (super)group of jets of diffeomorphisms of the neighborhood of the origin in $\mathbb{R}^{r|s}$ that fix the origin. Denote it $G(r|s)$. The usual general linear supergroup $GL(r|s)$ is the factor group of $G(r|s)$ w.r.t. the normal subgroup of jets of diffeomorphisms preserving the first infinitesimal neighborhood of the origin. It makes sense to speak about the Berezinian for such jets of diffeomorphisms, which is simply the Berezinian of the corresponding Jacobi matrix (i.e. the pull-back of the Berezinian on $GL(r|s)$). The covariance condition for an $r|s$-Lagrangian L means that

$$L(g[x]) = (\mathrm{Ber}\, g)^{\rho} \cdot L([x]) \tag{4.51}$$

for all $g \in G(r|s)$. This is a nonlinear analog of the covariance condition for the first-order Lagrangians [18].

The infinitesimal version of the condition (4.51) can be written as

$$(-1)^{\tilde{\sigma}\tilde{K}}\, \mathrm{D}_{\sigma}^{|\sigma|}(K^i(t)x_i^a)\frac{\partial L}{\partial x_{\sigma}^a} = \rho(-1)^{\tilde{\imath}(\tilde{K}+1)}\frac{\partial K^i(t)}{\partial t^i}L \tag{4.52}$$

at $t = 0$, where $K = K^i(t)(\partial/\partial t^i)$ is an arbitrary vector field vanishing at the origin. (Clearly, both sides depend only on its jet at the point 0.) Explicitly, the property (4.52) is equivalent to the following sequence of identities:

$$x_i^a\frac{\partial L}{\partial x_j^a} + 2x_{ik}^a\frac{\partial L}{\partial x_{jk}^a} + \cdots = \rho(-1)^{\tilde{\imath}}\delta_i^j L, \tag{4.53}$$

$$x_i^a\frac{\partial L}{\partial x_{jk}^a} + 3x_{il}^a\frac{\partial L}{\partial x_{jkl}^a} + \cdots = 0, \tag{4.54}$$

and so on to

$$x_i^a \frac{\partial L}{\partial x^a_{j_1\cdots j_N}} = 0, \tag{4.55}$$

where L is a covariant $r|s$-Lagrangian of weight ρ and order N.

If $K = K^i(t)(\partial/\partial t^i)$ is a genuine vector field on the space of parameters $\mathbb{R}^{r|s}$, not necessarily vanishing at 0, then it follows from (4.52) that for an arbitrary covariant $r|s$-Lagrangian L of weight ρ

$$K^{(\infty)}L + \rho(-1)^{\tilde{\imath}(\tilde{K}+1)}\frac{\partial K^i}{\partial t^i}L = 0, \tag{4.56}$$

where $K^{(\infty)} = K^i(t)\,\mathrm{D}_i - \mathcal{P}_{K^i x_i^a(\partial/\partial x^a)}$ is the prolongation of the vector field K (see Section 2.4).

Let us consider relations between covariant Lagrangians and differential forms.

In the case of ordinary purely even manifold M^m (odd dimension is equal to zero) one can assign to every differential r-form

$$\omega = \frac{1}{r!}\omega_{a_1\cdots a_r}\,\mathrm{d}x^{a_1}\cdots \mathrm{d}x^{a_r} \tag{4.57}$$

the following covariant r-Lagrangian (i.e. $r|0$-Lagrangian) of the first order and of weight $\rho = 1$:

$$L_\omega = \omega_{a_1\cdots a_r} x_1^{a_1}\cdots x_r^{a_r}. \tag{4.58}$$

For every r-path γ, $\int_\gamma L_\omega = \int_\gamma \omega$ and

$$L_{\mathrm{d}\omega} = \delta\!\!\!{}^{-} L_\omega, \tag{4.59}$$

where $\mathrm{d}\omega$ is the usual exterior differential of the form ω. Notice that under the identification of tangent elements with jets in $J^\infty(\pi_{(r)})$, the projection of the pull-back of an r-form ω to on $J^\infty(\pi_{(r)})$ onto the subspace $\Omega^{0,r}$ of horizontal r-forms is equal to the form $L_\omega(x)\mathrm{d}t^1\cdots \mathrm{d}t^r$.

In general, the variational differential $\delta\!\!\!{}^{-}$ does not take covariant Lagrangians to covariant Lagrangians and it increases the order: if L has order k, then, generally, the order of $\delta\!\!\!{}^{-} L$ is equal to $2k$.

Proposition 4.4. *If a Lagrangian L on a purely even manifold M is of the first order and the Lagrangian $\delta\!\!\!{}^{-} L$ is of the first order too, then the Lagrangian L is up to a constant a covariant Lagrangian corresponding to a differential form: $L = L_\omega + c$. The Lagrangian $\delta\!\!\!{}^{-} L$ corresponds to the differential form* $\mathrm{d}\omega$.

This proposition can be easily checked by straightforward calculations. It has the following geometrical meaning. In the space of paths on the manifold M consider a collection of topologies $\mathcal{T}_k$ $(k = 0, 1, \dots)$ such that sequence of paths γ_n tends to γ in the topology $\mathcal{T}_k$ if $|x_n(t) - x(t)| \to 0$ and a

$$\left|\frac{\partial^k x_n^a(t)}{\partial t^{i_1}\cdots\partial t^{i_l}} - \frac{\partial^k x^a(t)}{\partial t^{i_1}\cdots\partial t^{i_l}}\right| \to 0 \tag{4.60}$$

for all derivatives of order $l \leq k$. For a given Lagrangian L of order k the corresponding functional on paths is continuous in topology $\mathcal{T}_k$. On the other hand, if a first-order covariant Lagrangian L corresponds to a differential form: $L = L_\omega$, then due to the Stokes theorem the functional $S[\gamma]$ corresponding to this Lagrangian is continuous not only in the topology $\mathcal{T}_1$ but in the weaker topology $\mathcal{T}_0$. The converse implication leads to Proposition 4.4 (see [4,6,12]). In the supercase these considerations yield the definition of forms [15].

Definition 4.4. A covariant $r|s$-Lagrangian L of the first order on a supermanifold M is called an *$r|s$-form* if the Lagrangian $\text{đ}L$ is of the first order too.

The Lagrangian $\text{đ}L$ is $(r+1, s)$-form if L is an $r|s$-form and the Stokes theorem is valid. As follows from Proposition 4.4, on ordinary manifolds $r|0$-forms are in 1–1-correspondence with the usual differential r-forms and đ corresponds to the usual d, see, for details, [18].

Remark 4.9. The complete theory of forms in super case includes also objects defined similarly to Definition 4.4, but in the dual setting, using so-called copaths [17,19,20].

Consider now the action of the variational differential đ for covariant Lagrangians of higher order.

Proposition 4.5. *Covariant Lagrangians that are coboundaries in the complex of variational derivatives must be of the first order.*

Proof. Let a coboundary $\text{đ}L = x_r^a \mathcal{F}_a \in \Phi^{r|s}$ be a covariant $r|s$-Lagrangian. Then it must have weight $\rho = 1$, because it is linear in variables x_r^a. Let us show that $\text{đ}L$ is of the first order. Consider the identity (4.53) for the case when index i is equal to r. The variational derivative $\mathcal{F}_a$ does not depend on variables $x^a_{i_1\cdots i_k}$ if at least one of indices $i_1, \ldots, i_k$ is equal to r and $k \geq 2$. Hence, it follows from condition (4.53) that $\text{đ}L$ does not depend on variables $x^a_{i_1\cdots i_k}$ if $k \geq 2$, thus, the Lagrangian $\text{đ}L$ is of the first order. □

It follows from this proposition that a covariant closed $r|s$-Lagrangian L of order higher than one is a non-trivial cocycle of the complex of variational derivatives $\Phi^{*|s}$.

Finally, we consider some constructions for higher-order covariant Lagrangians.

1. *Lie derivative and the variational derivative of covariant Lagrangians*. Let $X = X^a(x)(\partial/\partial x^a)$ be an arbitrary vector field on M and L be a covariant $r|s$-Lagrangian of arbitrary weight ρ. Then the Lie derivative of this Lagrangian along the vector field X, $\mathcal{L}_X L = \mathcal{P}_X L = (-1)^{\tilde{\sigma}\tilde{a}}\, \mathrm{D}_\sigma^{|\sigma|} X^a(\partial L/\partial x_\sigma^a)$, obviously, is also a covariant $r|s$-Lagrangian of the same weight and of the same order.

   The Lagrangian L and the vector field $X = X^a(x)(\partial/\partial x^a)$ yield also another $r|s$-Lagrangian $X^a(x)\mathcal{F}_a(L)$, where $\mathcal{F}(L)$ is the variational derivative of the Lagrangian L (see (4.4)). The Lagrangians $\mathcal{L}_X L$ and $X^a(x)\mathcal{F}_a(L)$ differ by a divergence: $\mathcal{L}_X L - X^a(x)\mathcal{F}_a(L) = \mathrm{D}_i B^i$ (compare the end of Section 3.2). If the covariant Lagrangian L has weight $\rho = 1$, then one can prove that $X^a(x)\mathcal{F}_a(L)$ is also covariant of weight $\rho = 1$. In general, it has order $2k$, if the Lagrangian L has order k.

2. *Composition of Lagrangians*. Let L be an $r|s$-Lagrangian on a manifold N and let F be an $r|s$-Lagrangian on a manifold M taking values in the manifold N. Then one can consider the $r|s$-Lagrangian $L\circ F$ called the *composition* of these Lagrangians. If x^a and y^μ are local coordinates on M and N, respectively, $L = L([y])$ and $F : y^\mu = F^\mu([x])$, then consider the formal substitution

$$y^\mu = F^\mu([x]), \tag{4.61}$$

$$y_i^\mu = \mathrm{D}_i F^\mu([x]), \tag{4.62}$$

$$y_{ij}^\mu = \mathrm{D}_{ij}^2 F^\mu([x]), \tag{4.63}$$

and so on. The Lagrangian $L\circ F$ is obtained from L by this substitution.

One can see that if the Lagrangian L is closed, then the Lagrangian $L\circ F$ is also closed:

$$\bar{d}L = 0 \Rightarrow \bar{d}(L\circ F) = 0. \tag{4.64}$$

Notice the special case when F is a covariant $r|s$-Lagrangian of the weight $\rho = 0$. (F can be viewed as an N-valued function on jets of $r|s$-surfaces in M.) In this case, if L is a covariant (closed) $r|s$-Lagrangian of weight ρ, then $L\circ F$ is also covariant (closed) Lagrangian of the same weight ρ. The order of Lagrangian $L\circ F$ is equal, in general, to the sum of orders of the Lagrangians L and F.

Consider the following toy examples of this construction.

Let M be the Euclidean space $\mathbb{R}^m$. For every r, $1\le r\le m$ consider as the target space N the manifold of oriented r-dimensional linear subspaces of $\mathbb{R}^m$ (the oriented Grassmannian $G_r^+ = G_r^+(\mathbb{R}^m)$). Consider a function F_r with values in G_r^+ such that F_r assigns to every point $x\in\mathbb{R}^m$ and an arbitrary oriented r-dimensional plane Π^r through this point the oriented linear subspace parallel to Π^r. The function F_r defines on $\mathbb{R}^m$ a covariant r-Lagrangian of weight $\rho = 0$ and of order $k = 1$ with values in the oriented Grassmannian G_r^+.

Let ω be an arbitrary closed r-form on G_r^+ and let the Lagrangian L_ω correspond to ω (see (4.58)). Then the composition $L_\omega\circ F_m$ of Lagrangians L and F_m is a closed covariant Lagrangian of weight $\rho = 1$ and order $k = 2$. If forms ω are not cohomologous to zero we come to covariant Lagrangians $L_\omega\circ F$ corresponding to top-degree characteristic classes of surfaces embedded in $\mathbb{R}^n$. We shall consider from this point of view the Euler classes for r-dimensional surfaces embedded in $\mathbb{R}^m$ in two cases: $m = n-1$, $n = 2k+1$ (even-dimensional hypersurfaces in $\mathbb{R}^m$) and $m = 2$ (two-dimensional surface, embedded in an arbitrary $\mathbb{R}^m$).

Case 1 ($r = m-1$). The oriented Grassmannian G_{m-1}^+ is simply the sphere S^{m-1}. Let ω be a volume form on S^{m-1}:

$$\omega = \iota_E(r^{-m})\mathcal{D}x = \sum \frac{(-1)^{a-1}x^a\,\mathrm{d}x^1\cdots\widehat{\mathrm{d}x^a}\cdots\mathrm{d}x^m}{r^m}, \tag{4.65}$$

where $\mathcal{D}x$ is the standard coordinate volume form on $\mathbb{R}^m$, $E = X^a(\partial/\partial x^a)$ is the Euler field and $r^2 = (x^1)^2+\cdots+(x^m)^2$. Clearly, F_{m-1} supplies the Gauss spherical map for each oriented hypersurface $C^{m-1}\subset\mathbb{R}^m$. It is not difficult to see that the value of the covariant

Lagrangian $L = L_\omega \circ F_{m-1}$ gives the Gauss curvature density. In the case $m-1 = 2k$, $\int L$ is the Euler class.

Case 2 ($r = 2$). Let L be a covariant 2-Lagrangian in the Euclidean space $\mathbb{R}^m$ such that for every two-dimensional oriented surface $C \subset \mathbb{R}^m$,

$$\int_C L\mathcal{D}t = \int L\left(x\left(t, \frac{\partial x}{\partial t}, \dots\right)\right)\mathcal{D}t = \int_C R\sqrt{\det g}\,\mathcal{D}t, \tag{4.66}$$

where $x^a(t^i)$ is an arbitrary parameterization of the surface C ($i = 1, 2$), $g = (g_{ij})$ is the Riemannian metric induced on the surface C:

$$g_{ij} = \sum \frac{\partial x^a}{\partial t^i}\frac{\partial x^a}{\partial t^j}, \tag{4.67}$$

and R is the scalar curvature of this metric. Straightforward calculations show that

$$L = \sum_{a,b} \frac{(x^a_{11}x^b_{22} - x^a_{12}x^b_{12})P^{ab}}{\sqrt{\det g}}, \tag{4.68}$$

where

$$P^{ab} = \delta^{ab} - x^a_i g^{ij} x^b_j \tag{4.69}$$

is the projector on the plane orthogonal to the surface C, g^{ij} is the tensor inverse to the metric tensor g_{ij}.

We shall represent the covariant 2-Lagrangian (4.68) as the composition of the Lagrangians L_ω and F_2, where ω is a closed 2-form on Grassmannian G^+_2. Consider on the Grassmannian G^+_2 the homogeneous coordinates (u^a_i) ($a = 1, \dots, m$, $i = 1, 2$) such that u_i and u'_i define the same point (oriented 2-subspace of $\mathbb{R}^m$ spanned by the vectors u_1, u_2) if $u'_i = t^j_i u_j$, where t^i_j is a 2×2 matrix with the positive determinant. In these coordinates the covariant Lagrangian F_2 with values in the Grassmannian G^+_2 is $F_2(x^a_1, x^a_2) = (x^a_1, x^a_2)$. Thus the covariant 2-Lagrangian L in (4.68) is equal to $L_\omega \circ F_2$, where the 2-form ω on G^+_2 is given by the following formula:

$$\omega = \sum_{a,b} \frac{\mathrm{d}u^a_1\,\mathrm{d}u^b_2 P^{ab}(u_1, u_2)}{\sqrt{\det g(u_1, u_2)}}, \tag{4.70}$$

where $g(u_1, u_2)$ is the Gram matrix for the 2-frame u_1, u_2 and

$$\begin{aligned} P^{ab} &= \delta^{ab} - u^a_i g^{ij} u^b_j \\ &= \delta^{ab} - \frac{u^a_1 u^b_1 (u_2 \cdot u_2) - (u^a_1 u^b_2 + u^b_1 u^a_2)(u_1 \cdot u_2) + u^a_2 u^b_2 (u_1 \cdot u_1)}{\det g(u_1, u_2)} \end{aligned} \tag{4.71}$$

is projector on the plane orthogonal to the plane spanned by u_1, u_2.

The cohomology class of the closed 2-form (4.70) is a generator of the group $H^2(G^+_2(\mathbb{R}^m)) = \mathbb{R}$. To get a better understanding of this form consider the orthonormal homogeneous coordinates on $G_2^+(\mathbb{R}^m)$, i.e. the coordinates n_i^a of the vectors n_1, n_2 of an orthonormal basis of the corresponding subspace (i.e. a point of G_2^+). In these coordinates the form (4.70) has the appearance

$$\omega = \sum \mathrm{d}n_1^a \, \mathrm{d}n_2^a.$$

It seems plausible that constructions using composition of Lagrangians can be helpful for the study of topological invariants of surfaces. This approach can be naturally generalized to the supercase, where standard geometrical considerations are unavailable.

Acknowledgements

One of us (H.M.K.) is grateful to A. Verbovetsky for many inspiring discussions of jet geometry in 1998–1999. We would like to thank the referee for detailed remarks that helped to improve the original manuscript. The research was supported in part by EPSRC, under Grant GR/N00821/01.

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